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**OBSERVER DESIGN for
INTERCONNECTED SYSTEMS and
IMPLEMENTATION via
DIFFERENTIAL - ALGEBRAIC
EQUATIONS**

by

Barton Wassermann

To be presented to
The Open University
Milton Keynes
United Kingdom
for the degree of
Master of Philosophy
in
Mathematics

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Abstract

A new approach to the design of observers of nonlinear dynamical systems is presented. Generally, linear or nonlinear control systems are expressed as explicit systems of differential equations and solved either analytically or numerically. If numerically, they are implemented using standard ordinary differential equation (ODE) solvers. In this thesis, a system is decomposed and modeled as an interconnection between two observer subsystems, particularly, as canonical DAE observers. In general, control design engineers may be faced with a formidable problem of solving this system analytically or in obtaining closed-form solutions. To attest to the complexity and complications in treating a system of interconnected DAE observer systems, a scaled-down version of a publication on “Small-Gain Theorem” is included in the appendix for the reader’s perusal. (A brief introduction to “Small-Gain Theorem” can be found in Chapter 4). The premise of this thesis is to demonstrate that, where the design of an observer plays a major role involving output feedback, there may be advantages in formulating a control system as a differential-algebraic equation (DAE), especially in the case of interconnected subsystems. An implicit system of interconnected DAE observers is considered and shown implementable using an existing DAE solver, whose resolution allows one the capability of computing input and output bounds. This is based on fixed or variable timesteps within the operating interval of each subsystem to ensure input-output stability (IOS) and the observability property of the interconnected observer system. The observer design method is based on the extended linearization approach. The basic background is provided for the design process of an interconnected observer system using DAE. Note, the application of the new approach has not been considered previously for the case of an interconnected DAE observer system.

KEY WORDS relative to this thesis:

affine systems; backward difference formula (BDF); canonical DAE observer; differential-algebraic equations (DAE); DASSL; extended linearization; gain scheduling; interconnected systems; index of DAE; Jacobian; Laplace transform; linear systems; local stability; lo-pass filter; Luenberger's systems; Lyapunov stability theory; optimal control; noise sensitivity analysis; MATLAB & Simulink; nonlinear control systems; observer design; small-gain theorem; system stability

Contents

1	Introduction	1
1.1	Organization of Thesis	1
1.2	Observer Design Problem	4
1.3	Observers	7
1.4	Reduced-Order Observers	22
1.5	Systems with Reduced-Order Observer	26
1.6	Example	29
2	Formulation of DAE Observer	38
2.1	DAE Observer System	38
2.2	Index of DAE	39
2.3	Canonical DAE Observer	43
2.4	Convergence Issue	47
2.5	Sensitivity to Noise	50
2.5.1	Standard approach	52

2.5.2	Noise sensitivity analysis	55
2.5.3	Canonical DAE observer approach	58
2.5.4	Conclusion	59
3	Estimation Error of Canonical DAE Observer	63
3.1	Error Analysis	63
3.2	Local Stability	65
3.3	Extended Linearization	72
4	Interconnected DAE Observers	77
4.1	Interconnected Systems	77
4.2	Feedback Interconnection System	85
4.3	Interconnection DAE System Stability	97
4.4	Small-Gain Theorem	99
4.5	Numerical Implementation	114
4.6	BDF Convergence	116
4.7	Implementation Considerations	118
4.8	DAE Solvers in MATLAB and Simulink	123
4.9	Analytical Solutions	130
5	Results	153
A	Optimal Control	160
A.1	Problem Statement	160

A.2 Conversion to General Statement	162
A.3 Solution	163
A.4 Tractability Index	165
 B Laplace Transformation	 167
B.1 Partial List of Laplace Transform Pairs	168
B.2 Properties of Laplace Transforms	168
B.3 Example	169
 C Observers	 170
C.1 Reduced-Order Observers	170
 D Small-Gain Theorem	 180
D.1 Definitions and Theorem	180
 E List of Symbols & Notation	 186
 F MATLAB & Simulink Codes	 189

List of Figures

1.1	A linear feedback control system with a compensator	17
1.2	Observer structure for generating a state estimate	21
1.3	System model of a 1-dimensional tracking problem where x_1 and x_2 are position and velocity of tracked object, respectively.	34
1.4	Observer design of tracking example in Figure 1.3 using trans- formation as described in Section 1.6	35
1.5	Graph of state variables $x_1(t)$ (actual) and $\hat{x}_1(t)$ (estimated) in tracking example. $\hat{x}_1(t)$ estimates $x_1(t)$ exactly for time t . That is, $\hat{x}_1(t) = x_1(t) = 3 - 2e^{-t} - t$	36
1.6	Graph of state variables $x_2(t)$ (actual) and $\hat{x}_2(t)$ (estimated) in tracking example. $\hat{x}_2(t)$ estimates $x_2(t)$ closely for time t , where $\hat{x}_2(t) = -1 + 2e^{-t} - e^{-5t}$ and $x_2(t) = -1 + 2e^{-t}$	37
2.1	Graph of error estimates for full - order observer case	60
2.2	Graph of error estimates for reduced - order observer case . . .	61
2.3	Graph of error estimates for canonical DAE observer case . . .	62
4.1	Block diagram of each control system Σ_1, Σ_2	85

4.2	Σ - The interconnected system	86
4.3	Interconnected canonical DAE observer design	87
4.4	Interconnected canonical DAE observer compensators	94
4.5	Σ - The interconnected system for similar model design	97
4.6	Interconnected canonical DAE observer for similar model design	98
4.7	Internal Stability	103
4.8	Graphs for ODE (4.5)-(4.7) (subsystem Σ_1), for input $u_1 =$.65, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0,$ $x_2(0) = 1.1, y_1(0, u_1) = -k_1 u_1$	135
4.9	Graphs for ODE (4.8)-(4.10) (subsystem Σ_2), for input $u_2 =$.7, parameters $k_2 = [-1.2, 1.2]$, initial conditions $x_1(0) =$ $-1.1, x_2(0) = 0, y_2(0, u_2) = -k_2 u_2$	136
4.10	Graphs for ODE (4.5)-(4.7) (subsystem Σ_1), for input $u_1 =$ 1.5, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0,$ $x_2(0) = 1.1, y_1(0, u_1) = -k_1 u_1$	137
4.11	Graphs for ODE (4.8)-(4.10) (subsystem Σ_2), for input $u_2 =$ 1.5, parameters $k_2 = [-.3, .3]$, initial conditions $x_1(0) = -1.1,$ $x_2(0) = 0, y_2(0, u_2) = -k_2 u_2$	138
4.12	Graphs for ODE (4.5)-(4.7) (subsystem Σ_1), for input $u_1 = 2,$ parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0, x_2(0) =$ 1.1, $y_1(0, u_1) = -k_1 u_1$	139
4.13	Graphs for ODE (4.8)-(4.10) (subsystem Σ_2), for input $u_2 =$ 2, parameters $k_2 = [-.5, .5]$, initial conditions $x_1(0) = -1.1,$ $x_2(0) = 0, y_2(0, u_2) = -k_2 u_2$	140
4.14	Graphs for DAE observer (4.23) (subsystem Σ_1), for input $u_1 = .65$, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) =$ 0, $x_2(0) = 1.1, y_1(0, u_1) = -k_1 u_1$	141

4.15	Graphs for DAE observer (4.24) (subsystem \sum_2), for input $u_2 = .7$, parameters $k_2 = [-1.2, 1.2]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$, $y_2(0, u_2) = -k_2 u_2$	142
4.16	Graphs for DAE observer (4.23) (subsystem \sum_1), for input $u_1 = 1.2$, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$, $y_1(0, u_1) = -k_1 u_1$	143
4.17	Graphs for DAE observer (4.24) (subsystem \sum_2), for input $u_2 = 1.5$, parameters $k_2 = [-.6, .6]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$, $y_2(0, u_2) = -k_2 u_2$	144
4.18	Graphs for DAE observer (4.23) (subsystem \sum_1), for input $u_1 = 2$, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$, $y_1(0, u_1) = -k_1 u_1$	145
4.19	Graphs for DAE observer (4.24) (subsystem \sum_2), for input $u_2 = 2$, parameters $k_2 = [-.3, .3]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$, $y_2(0, u_2) = -k_2 u_2$	146
4.20	Graphs: DAE estimated state variables $\hat{x}_1^{(1)}$, $\hat{x}_2^{(1)}$ and output \hat{y}_1 converge to ODE results.	147
4.21	Graphs: DAE estimated state variables $\hat{x}_1^{(2)}$, $\hat{x}_2^{(2)}$ and output \hat{y}_2 converge to ODE results.	148
4.22	\sum_1 Subsystem: Comparing the plot of analytical solution (4.32) with ODE numerical solution to 4.5, 4.6.	149
4.23	\sum_2 Subsystem: Comparing the plot of analytical solution (4.33) with ODE numerical solution to 4.8, 4.9.	150
4.24	\sum_1 Subsystem: Comparing the plot of analytical solution (4.32) with DAE numerical solution to 4.23.	151
4.25	\sum_1 Subsystem: Comparing the plot of analytical solution (4.33) with DAE numerical solution to 4.24.	152

F.1	Graphs of solutions to DAE observer (4.23),(4.24) for system Σ , using an ODE solver for input $u_1 = 1, u_2 = 1$, parameters $k_1 = [-.3, .3], k_2 = [-.3, -.3]$, initial conditions $x_1^{(1)}(0) = 0, x_2^{(1)}(0) = 1.1, y_1(0, u_1) = -k_1 u_1, x_1^{(2)}(0) = 0, x_2^{(2)}(0) = -1.1, y_2(0, u_2) = -k_2 u_2$	196
F.2	Solutions to DAE observer (4.23),(4.24) for system Σ , using an DAE solver for input $u_1 = 1, u_2 = 1$, parameters $k_1 = [-.3, .3], k_2 = [-.6, .6]$, initial conditions $x_1^{(1)}(0) = 0, x_2^{(1)}(0) = 1.1, y_1(0, u_1) = -k_1 u_1, x_1^{(2)}(0) = 0, x_2^{(2)}(0) = -1.1, y_2(0, u_2) = -k_2 u_2$	203

Chapter 1

Introduction

1.1 Organization of Thesis

The objective of this thesis is to show that for control systems expressed as explicit systems of differential equations, there can be advantages in formulating observer designs as implicit (descriptor) systems which can be numerically integrated using a differential-algebraic equation (DAE) solver. To this end, this thesis is organized in the following order:

In this Chapter, observer design problems relative to control systems are discussed. The error estimate of states, in terms of a linear time-invariant system, is discussed. Reduced-order observers are discussed, illustrated and

constructed, in order to discuss the similarity with respect to the formulation of exact measurements of observer output, y , of reduced-order observers and canonical DAE observers (see chapter 2). Control systems characterized by affine equations are discussed. An example is also given to demonstrate the construction of a reduced-order observer of a simplified tracking system. Section 1.3 discusses a historical view of observer designs which includes a recent reference on the evolution and development of structures of observers.

Chapter 2 contains pertinent information regarding the development of the canonical DAE observer and its corresponding state vector estimator. Some basic DAE theory including the definition of "index" of a DAE is described. The issue of convergence of \hat{x} , an estimated state, to the unknown state, x , for a DAE observer design and the advantageous application of Lyapunov Stability Theory in performing error analysis of state estimator are discussed. It is shown that a canonical DAE observer may have a linearizing effect on its DAE observer design, and, yet may not be linear for standard ODE observers. It also may lead to a linear error equation that is more tangible than that obtained by using standard observers. This could be a major advantage of the new approach. High frequency noise is of major concern since differentiation of algebraic equations can cause the solutions to become very sensitive to noise. Thus, we discuss three types of observer designs, namely, a full-order observer, a reduced-order observer, and the canonical DAE observer approach, as to their sensitivity to noise by employing an example to demonstrate the effect of high-frequency noise in each observer design. That is, an example is given and the different types of state observers (discussed in appropriate sections) are formulated. Then, a comparison analysis of the effect of high-frequency noise in each observer design is given. This study confirms that index one canonical DAE observers do not have the problem

of sensitivity to high-frequency noise that other observer designs have.

In Chapter 3, the analysis of estimation error of states associated with the canonical DAE observer is discussed. The local stability around equilibrium points is studied. The concept of constructing Hurwitz matrices is introduced here. The stability criteria related to extended linearization of canonical DAE observer are studied.

Chapter 4 extends the study of canonical DAE observers to interconnected DAE observer systems. The Small-Gain Theorem for interconnected systems is briefly discussed and a publication regarding this has been included in the appendix in order to demonstrate the complexity and dilemma an observer designer is faced with in using an analytical approach to solve design problems by ODE means. However, with the availability of DAE solvers, it may be advantageous to construct the interconnected observers as interconnected canonical DAE observers to reconstruct observer estimators. In this thesis, interconnected subsystems are reconstructed or modeled as interconnected canonical DAE observers which are designed and implemented in a format required by a DAE solver. The concept and mathematical development justifying the implementation of control systems are presented here. It is hoped that this methodology is accepted as a standard means in designing future interconnected observers. A methodology used in computing equilibrium points is described. An example is considered where an observer design of an interconnected feedback system is decomposed and modeled as an interconnected DAE observer system. Computation of the appropriate eigenvalues guaranteeing the stability of the interconnected system is shown. Numerical implementation and considerations are discussed. The implicit DAE observer satisfying certain conditions, is formulated to make it numerically integrable when using software for solving DAE problems. Also, in Chapter 4 we derive

analytical solutions to the examples of ODE second-order subsystems used in constructing interconnected DAE observer design. Graphs of analytical solutions are compared with those from a numerical integrator (ODE solver). It should be noted that this chapter contains a major portion of the author's original work. However, much of the documented theory and research, and author's detailed discussions included in most of the chapters, in particular Chapter 2, provide the background material required for the comprehension and development of this chapter.

Chapter 5 presents a synopsis and results of this research project. Suggestions and topics are provided so that further contribution can be made by researchers to the study of automatic control theory, particularly, on the subject of observer design and implementation via DAE.

The appendices contain additional information relative to various references indicated in the thesis. They consist of, for example, detailed background information which is beyond the scope of the thesis but may help understanding the contents being discussed. A partial list of the symbols and notation used in this thesis is also contained in the appendices. Appendix F contains samples of MATLAB codes used to produce graphics in Chapter 4.

1.2 Observer Design Problem

A multivariable nonlinear system with m inputs u and p outputs y described for $t > t_0$ by the dynamical equations

$$\dot{x} = f(x, u) \quad (\text{state equation}) \quad (1.1)$$

$$y = h(x) \quad (\text{output equation}) \quad (1.2)$$

is of a type for which the problem of “observer design” is considered in this thesis. The state x is assumed to belong to an open set $X \subseteq \mathbb{R}^n$, input u belongs to an open set $U \subseteq \mathbb{R}^m$, and output y belongs to an open set $Y \subseteq \mathbb{R}^p$.

The dynamical equations (1.1) and (1.2) describe the unique relation between the input, output, and state. It is assumed that f, h are \mathcal{C}^2 (continuous functions with continuous first and second derivatives) vector fields on \mathbb{R}^n and \mathbb{R}^p and that the $p \times n$ ($p < n$) Jacobian matrix $\partial h / \partial x$ has full rank for $x \in X$, i.e. p measurements in (1.2) are linearly independent.

The equations (1.1) and (1.2) represent a control system or process which has associated with it a set of input data as a $(m \times 1)$ vector $u = [u_1, u_2, \dots, u_m]^T$ and a set of output data as a $(p \times 1)$ vector $y = [y_1, y_2, \dots, y_p]^T$. The input and output vectors are assumed to be functions of time. The initial state of the system $x(t_0)$ at time t_0 , given input $u(t)$, $t > t_0$, uniquely determines the behavior (output and state) of the system $\forall t \geq t_0$. Note the state variables in $(n \times 1)$ vector representation is $x = [x_1, x_2, \dots, x_n]^T$.

The problem of observer design is to find a scheme for reconstructing the state x (or an estimate for x , \hat{x}) based on past and present (known and/or measured) values of the input u and the output y . In this thesis, the observer is formulated as an implicit system which can then be solved using a DAE solver. However, since in a later chapter we study the asymptotic stability of interconnected subsystems, we consider the following nonlinear dynamical system

$$\dot{x} = f(x, u)$$

$$y = \bar{h}(x, u)$$

where the input u enters into the equation (1.2) for h . The dimensions are as given for (1.1) and (1.2), $f(x, u)$ and $\bar{h}(x, u)$ are locally Lipschitz.

In real analysis a real function f defined on a subset D of real numbers $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be Lipschitz continuous if there exists a constant $K \geq 0$ such that for all $x_1, x_2 \in D$, $|f(x_1) - f(x_2)| \leq K |x_1 - x_2|$, where the smallest such K is called the Lipschitz constant of the function f . The function f is locally Lipschitz continuous if for every $x \in D$ there is a neighborhood $U(x)$ so that f restricted to U is Lipschitz continuous.

It should be clear that systems modeled by the above equations are not necessarily affine in the control input. Affine systems are represented by equations of the form

$$\dot{x} = \bar{f}(x) + g(x)u$$

$$y = h(x) + \bar{k}(x)u,$$

where $f(x, u)$, $\bar{h}(x, u)$ are characterized by the above equations. Note: g and \bar{k} have dimensions $n \times m$ and $p \times m$, respectively.

However, dynamical systems of this type will be of prime consideration in later sections, where $\bar{k}(x) = k$ ($\neq 0$) is a constant matrix.

It should be borne in mind throughout this thesis that even though the functions, $\bar{f}(x)$, and $h(x)$, represent different functions to $f(x, u)$ and $\bar{h}(x, u)$, respectively, it will be clear from the arguments which representation is meant. $\bar{k}(x) = k$ is considered a nonzero constant matrix in much of what follows, particularly, in Chapter 4.

For f differentiable, $\frac{\partial f}{\partial x} = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$ is the Jacobian matrix for $\bar{f}(x)$ or $f(x, u)$, or in familiar mathematical notation, $\bar{J}(x)$ or $J(x, u)$ for Jacobian matrices.

Note, when $f(x, u) = \bar{f}(x) + g(x)u$, we write $J(x, u) = \frac{\partial f(x, u)}{\partial x} = \frac{\partial \bar{f}(x)}{\partial x} + \frac{\partial g(x)}{\partial x}u$.

1.3 Observers

Observer Design:

In reality there is no systematic method to design an observer for a given nonlinear control system, but several designs are available according to specific characteristics of the considered system.

The concept of observer design engineering possesses at least fifty years of history of continuous research and development in the area of automatic control technology. Thus, to claim any one methodology as the state-of-the-art would be a meaningless and futile task. In many cases, if a linear/nonlinear control system does not completely satisfy any of the known properties, it may satisfy some of them partly. This is the basic reason for interconnections between several subsystems, where each of the subsystems satisfies some required properties for an observer to be computable.

A methodology for one observer design case may not be a suitable for another case. An automatic control engineer is, generally, equipped with a high-level background in applied mathematics in order to pursue research and development of a suitable observer design which may be the answer to his problem. For linear systems, the achieved controller based on the estimates of state given by the achieved observer is proved to be still stabilizing. However, this is no longer guaranteed for general nonlinear systems.

In reference [25], the authors present a historical view for the benefit of control engineers. This paper is helpful in the sense that all observers are examined in terms of: 1) assumed dynamic structure of the plant, 2) the required information, including the input signals and modeling information of the plant, and 3) the implementation equation of the observer.

This paper describes the evolution of research and development of structures of observers which have become integral part of control theory and engineer-

ing practice. Formulations of observer designs and mathematical models with associated implementation techniques are given as a guideline in providing an optimum solution to the physical problem at hand. These observer designs developed during the half century are listed in this paper. Estimators, such as linear/nonlinear Luenberger observers are also discussed. However, it should be noted that the authors point out that the Luenberger observers established the structure that most estimators are based on today. The difference lies in the method of choosing a gain matrix. Luenberger observers are covered in later sections in this thesis which deal with reduced-order observers. This paper also considers basic nonlinear observers which is also described in this thesis. An extensive reference list of the authors' source of information on which their paper is based, is provided.

Literature bears out that no one specific methodology can serve as the state-of-the-art in observer design. Very few recent publications exist on studies of DAE solutions to observer design problems of nonlinear control systems. A formal background in nonlinear observer systems can be gained from a textbook such as [9] (in Chapters 10 - 13), which provides a number of analytical approaches still very much in use to this day.

In this thesis, reconstructing observer designs using canonical DAE observers of index one is emphasized, particularly, in the case of two interconnected subsystems as developed in Chapter 4.

Based on many hours expended on research of recent publications relative to observer designs for interconnected systems and implementation by means of differential-algebraic equations (DAE), it is easily seen that this is potentially an unexplored area. A list of at least a few hundred abstracts has been compiled, but a pared down list of references (from [26] and above) is included in the bibliography for the reader's interest.

The following describes the basic contents of some the references:

[26] gives a simple convergence theorem on the waveform relaxation (WR) solution for a general nonlinear DAE circuit system of index one. Examples are given to confirm the theoretical analysis. In this thesis, we are concerned with the design of an observer in order to estimate unknown state variables and maintain a viable interconnected control system.

[27] deals with local observability with respect to nonlinear DAEs near a known trajectory associated with a given control. It establishes some sufficient conditions for local observability of nonlinear DAE systems near a known trajectory. This result overcomes the inadequacy of the time-invariant linearization for determining observability of nonlinear DAEs. Although it discusses DAE based on [3] (which is also covered in this thesis), it does not consider an observer design for interconnected DAE subsystems (see Chapter 4 in this thesis) from an interacting system given as an ODE model which is then transformed into an interconnected canonical DAE observer system. The local stability of each interconnecting subsystem is determined and the stability of the entire system is then established. However, it is conceivable that for some nonlinear canonical observers, the results of this paper with regards to interconnected subsystems may be a worthwhile research effort to pursue in the future.

[28] presents the clustering approach to calculate the limits of parametric stability for a stable equilibrium of the power systems modeled by differential algebraic equations (DAEs), called the feasibility boundary. The underlying algorithm calculation is based on the sensitivities of the eigenvalues of DA system. Several examples are presented. This paper demonstrates a strong applicability in solving an electric power system, which may be modeled by

a DAE. Although, this paper is not relevant to my thesis per se, it does appear that the subject of interconnected subsystems modeled as canonical DAE observers may be appropriate for this application.

[29] is basically the same subject matter as in [26], but develops and unifies convergence results in WR iteration for a general class of DAEs. That is, a new convergence theory on WR for nonlinear DAEs of index one, which can not only analyze WR solutions of a rather general system, but can also unify known WR convergence results. This theory may also be a basis of higher index DAEs on WR. Its nonrelevance to control problems is as stated for reference [26] above.

[30] analyzes the numerical behavior of multi-step integration formulas for solving DAEs with higher index. This is applied to electronic circuits modeled by DAEs. This paper shows that the class of Gear's backward differentiation formula, unlike other multi-step techniques, are useful means for obtaining consistent initial conditions when carefully implemented. The numerical experiments suggest that the method works reliably for index-3 DAEs. In this thesis, DAEs of index of no higher than one is considered.

[31] considers the simplest stationary linear differential-algebraic systems of observation and control with delays. This paper is not relevant to this thesis.

[32] considers singular systems of nonlinear DAEs whose constrained state-space depends on control inputs. A state-space realization of such systems cannot be derived independently of the controller design. An output feedback pre-compensator is derived, which results in a modified DAE system whose state-space is invariant under any feed back control law and can be used for output feedback synthesis. A nonlinear electrical circuit is given as an example. In this thesis, the inputs of each plant in the negative feed-

back interconnected observer system, is provided by the output feedback pre-compensator to the plants (see Figure 4.5). This paper considers nonlinear DAE systems as semi-explicit descriptor which may be of high index whereas this thesis canonical DAE observers of index one where interconnected at the input level.

[33] considers the problem of designing global state observer for a class of nonlinear systems based on input-output linearization. it proposes a procedure for the design of nonlinear state observers which do not require the hypothesis of full relative degree. It deals with a class of systems which can be transformed into a global normal form in which the internal dynamics satisfied some Lyapunov-like conditions. For the local observation, exponential strability of zero dynamics is sufficient to guarantee that the output of the proposed observer converges to true state. Much of the discussion in his approach applies an area of mathematics, differential geometric (Lie Algebra) which can be found [9]. It is interesting to note that this paper considers single input - single output (SISO) nonlinear systems of the form $\dot{x} = f(x) + g(x)u = f(x, u)$, $y = h(x)$ whereas, in this thesis the affine form $\dot{x} = f(x) + g(x)u = f(x, u)$, $y = h(x) + k(x)u = \bar{h}(x, u)$ is considered.

[34] addresses the problem of transforming nonlinear system into nonlinear observer canonical form in the extended state-space with aid of dynamic system extension introduction of virtual outputs. The paper, then proposes a restricted class of dynamic extension, which can be obtained by adding chains of integrators or special linear systems to the output. Sufficient conditions are provided for dynamic observer error linearization for the restricted structure. In this thesis, we transform nonlinear observer design system expressed in ODE into the canonical DAE observer. We study the error estimates of state variables and determine eigenvalues which ensure matrix, $\Gamma(\hat{x}, u)$, to

be Hurwitz (see Section 3.3 on extended linearization). We, also, study local stability of error estimates state variables at or near an equilibrium point. These facts are used in computing solutions to canonical DAE observers of index one via DAE solver.

[35] studies observer design for feedback regulation in unsteady fluid flows. This paper estimates and feed-back-regulates the location of the vorticity centroid and the mutual distance of an interacting, same sign vortex pair. The working model for observer design is that of constant rotation, and the observer is an extended Kalman filter (EKF). Certain advantages include simplicity, robustness, and the filtering of unmodeled dynamics. Note, EKF is covered in [25]; however, the approach may or may not be suitable for nonlinear index one observer design systems.

[37] addresses the problem of observer design for a class of Lipschitz nonlinear discrete-time systems. By some simple transformation, both full-order and reduced-order state observers are established. From particular Lyapunov functions, weak sufficient conditions for asymptotic stability are provided in terms of linear matrix inequality (LMI). Performances of the proposed approach are illustrated through simulation and experimented results; one of these concerns synchronization of chaotic nonlinear models. Note, that although the discrete-time systems are not covered in this thesis, this Chapter discusses full-order and reduced-order observers for the continuous time case in this chapter. These can easily be transformed to the discrete-time case.

[38] formulates connective stability conditions for discontinuous interconnected systems in terms of both Lipschitz and C^1 -type vector Lyapunov functions. Vector Lyapunov functions are parameter dependent due to polytopic structure of interconnection terms regarded as perturbations of the subsystem dynamics. Connective stability conditions are formulated as con-

vex optimization problems relying on M-matrix theory. Therefore, if the problem is feasible, convex programming theory guarantees that a solution will be computed. Finally, using results on connective stability, it is shown that the generalized matching conditions for stabilizability are also valid for piece-wise continuous type discontinuities. Although this paper is not relevant to this thesis as the problem is that of feedback control which depends on the output of the connecting compensator.

A brief introductory background to this paper is very well described in reference [9] in a section called interconnected systems for a mathematical model of an artificial neural network. However, this paper formulates connection stability as a convex optimization problem and applies an efficient convex programming tool such as, linear programming and convex algorithms for linear matrix inequalities which is not relevant to this thesis. This thesis is concerned with the formulation of DAE observers for two interconnected control subsystem and solving the dynamic equations by means a DAE solver for DAE systems of index one available with MATLAB & Simulink software. Perhaps, future research should entail studying multi-connected subsystems such as the above proposes.

In [39], the problem of robust decentralized control for a class of uncertain nonlinear interconnected systems with mismatched uncertainties is investigated. The proposed state feedback controllers ensure that the closed loop state is globally exponentially stable. An example is presented to illustrate the results given in this paper and simulation shows that the method is effective.

Note, the above paper is published in Chinese except for the abstract.

In paper [40], using decentralized control method, passivity property of interconnected control systems with time-delays is studied. Both existence results

and expressions of decentralized and passive controller are presented. The conditions are represented by linear matrix inequality (LMI). The authors claim the results can be implemented easily in practice.

As for the term passive approach, it is an alternative approach to the stability analysis of feedback systems. For example, if in the feedback connection of Figure 4.5 (Chapter 4), both subsystems G_1 , G_2 are passive in the sense that they do not generate energy of their own, then the feedback system will be passive. However, if one of the two feedback components dissipates energy, then the system will dissipate energy.

The purpose of this paper is to design a local memoryless state feedback controller, guaranteeing that the closed-loop system is stable and strictly passive. Note, memoryless functions is a type of passive system. This paper uses the energy concept which suggests constructing a Lyapunov functional to solve the problem. This thesis is not concerned with the approach above; however, it is conceivable that a canonical DAE observer of index one could be applied to the interconnected control system with time-delays as proposed to establish system stability.

[41] considers the problem of decentralized robust controller for a class of uncertain large scale interconnected systems. An adaptive sliding mode controller for an uncertain large scale interconnected system with unknown bounds of uncertainties and unknown degrees of interconnection is proposed. A numerical example is given to demonstrate the validity of the results.

[43] presents the extension of the scaled Small Gain Theorem. The scaled Small Gain Theorem gives necessary and sufficient conditions for robust stability of a nominal linear time-invariant system in feedback with structural operators of norm less than or equal to unity. An alternative linear matrix inequality conditions that give necessary and necessary conditions for robust

stability against the class of structural unitary operators (invertible operators of exactly unit norm) is proposed. It is shown that this result, besides being a less conservative version of the scaled small gain theorem, has connections to several recent results on the control of spatially interconnected system and serves to unify and quantify the conservatism of those results. Small Gain Theorem is presented in Chapter 4 in this thesis for a different purpose; however, this does not include spatially interconnected systems which are taken over a hexagonal array. This paper is not concerned with such observer design systems. However, I believe that the interconnected canonical DAE observers would be an appropriate area to investigate with respect to this paper. This thesis could be a prelude to constructing observer designs relative to the interconnecting systems studied in this paper.

[36], [42] and [44] contain elements of fuzzy control and modeling which basically deal with the probabilistic aspect applied to known methodologies in control theory and modeling. Note, most of the subject matter is covered in other references discussed.

Note, the state-of-the-art for nonlinear observer design cannot be concluded from this reference except for the mixed methodologies presented in various textbooks as indicated here.

The Following is a Discussion on Observers:

Observability involves the effect of the state vector $x(t)$ on the output $y(t)$ of the linear state equations with input $u(t)$. For example, a linear system (a classic example as in (1.3), (1.4) below) is said to be observable at t_0 if $x(t_0)$ can be determined from the output function $y(t)$, $t \in [t_0, t_1]$, where t_1 is some finite time. If this is true for all t_0 and $x(t_0)$, the system is completely observable. If the system is not observable then the initial state $x(t_0)$ cannot

be determined from the output, no matter how long the output is observed (formal theorems can be found in [23]).

Some brief statements on feedback systems may be in order: 1) Feedback control is an operation which, in the presence of disturbances, tends to reduce the difference between the output of the system and the reference (or an arbitrarily varied, desired state), $r(t)$ (see Figure 1.1), and which does so on the basis of this difference. 2) A Feedback control system is one which tends to maintain a prescribed relationship between the output and the reference, $r(t)$, by comparing these and using the difference as a means of control. Generally, feedback control system designs (see Figure 1.1, for example) are based on the assumption that the state vector, $x(t)$, of the system to be controlled, is available for measurement at some point t . However, in many cases the entire state vector cannot be measured, as is typical in most complex systems. In Luenberger's papers, [10] and [11], it has been shown that the state vector of a linear system can be constructed from observations of the system inputs and outputs. Kailath in his book [8] presents this approach as well. In order to resolve the problem of controlling a system, an input vector $u(t)$ must be chosen with some scheme so that the system behaves in an acceptable manner. One may design a feedback law of the form, $u(t) = \Theta[x(t), t]$, where the input vector $u(t)$ is based on $x(t)$ and t . Since this relation contains all the essential information about the system, the system inputs and outputs are used to reconstruct the system state vector. But, if the entire state vector is not available for direct measurement, then it is not possible to evaluate the function $\Theta[x(t), t]$. However, a good estimate of the state vector may be obtainable given the knowledge of the inputs and outputs. To this end, another dynamic system, called an **observer** needs to be constructed. This observer which reconstructs the state vector $x(t)$ based on u and its measured/known

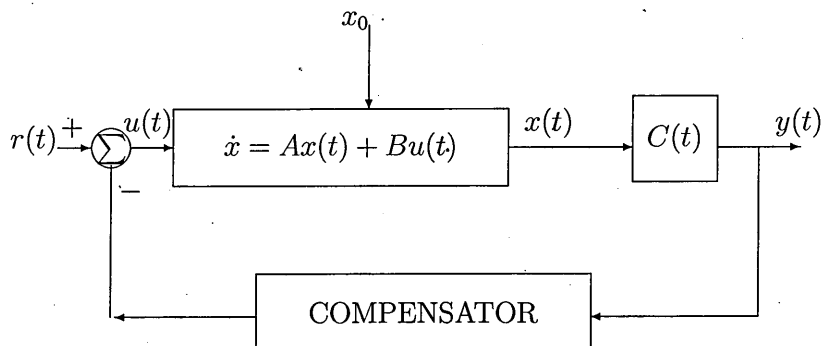


Figure 1.1: A linear feedback control system with a compensator

output y , yields a good approximation $\hat{x}(t)$ to $x(t)$. Further discussion on observers is found in this Section (see (1.6),(1.7), for an example of an observer)). In engineering literature, the term **state estimator** is sometimes used to mean a device that constructs an approximation of the state vector i.e. an observer. Note, for system (1.1), (1.2), the feedback law may be of the form $u(t) = \Psi[y(t)]$, $y(t) = h(x(t))$.

In Figure 1.1, we note a **COMPENSATOR** block in the resulting feedback path. Such a feedback control system is referred to as a feedback compensation or parallel compensation. In building a control system, we know that proper modification of the plant (e.g., the first block in the forward path of Figure 1.1) dynamics may be a simple way to meet performance specifications. This, however, may not be possible in many practical situations because the plant may be fixed and may not be modified. Then we must adjust parameters other than those in the fixed plant. The subject is introduced

here as observers are being discussed here; also “reduced-order observers” and “canonical DAE observers” will be discussed in the appropriate Sections.

Feedback compensation is one of several types of compensator that may be required in building a control system. It may also be necessary to employ a pole-assignment technique (process for placing eigenvalues) to construct an asymptotic stable observer such as (1.6), (1.7).

In a later paragraph, the observer reconstruction is discussed. See Section 1.6 for an example of this reconstruction.

The following is a brief discussion describing state observation in terms of linear system theory using state feedback via observers. Observers in turn play an important function in control problems involving output feedback. State observation involves using current and past values of a physical object to be controlled without the feedback connection equation (e.g., see first block in Figure 1.2), where the input and output signals are used to generate an estimate of the current state (assumed unknown). Consider the linear dynamical equations,

$$\dot{x}(t) = f(x, u) = A(t)x(t) + B(t)u(t) \quad (1.3)$$

$$y = h(x) = C(t)x(t) \quad (1.4)$$

where $x = x(t)$, with the initial state $x(t_0) = x_0$ unknown. We generate an $n \times 1$ vector $\hat{x}(t)$ that is an estimate of $x(t)$ in the sense that

$$\lim_{t \rightarrow \infty} [\hat{x}(t) - x(t)] = 0 \quad (1.5)$$

The $n \times 1$ vector function of time, $x(t)$, is a state vector with components, $x_1(t), \dots, x_n(t)$, called state variables. The input signal is a $m \times 1$ function $u(t)$, and $y(t)$ is a $p \times 1$ output signal, where $p, m \leq n$. The entries of $A(t)$ ($n \times n$), $B(t)$ ($n \times m$), and $C(t)$ ($p \times n$) are real valued functions of t

(assumed known). It is assumed that the procedure for producing $\hat{x}(t_a)$ at any $t_a \geq t_0$ can make use of the values of $u(t)$ and $y(t)$ for $t \in [t_0, t_a]$, as well as knowledge of the coefficient matrices in (1.3) and (1.4).

If system (1.3), (1.4) is observable on $[t_0, t_b]$, then one might instinctively attempt to obtain a state estimate, $\hat{x}(t)$, by first computing the initial state \hat{x}_0 from the knowledge of $u(t)$ and $y(t)$ for $t \in [t_0, t_b]$. Then solve (1.3) for $t \geq t_0$, yielding an estimate that is exact at any $t \geq t_0$, though not current. That is, the estimate is delayed because of the wait until t_b , the time required to compute x_0 , and then the time to compute the current state from this information.

From here on we assume $A(t) = A$, $B(t) = B$, $C(t) = C$, i.e. A, B, C as constant matrices in equations (1.3), (1.4).

Now, we generate an asymptotic estimate of (1.3) by using another linear state equation, say,

$$\dot{\hat{x}} = \bar{F}\hat{x}(t) + Gu(t) + Hy(t), \quad \hat{x}(t_0) = \hat{x}_0$$

with the property that (1.5) holds for any initial states x_0 and \hat{x}_0 , for some constant matrices \bar{F} , G and H that accepts input and output signals, $u(t)$ and $y(t)$, and we also choose $\bar{F} = A - HC$, $G = B$ (H is explained in a subsequent paragraph). This becomes our observer structure yielding a state estimate as diagrammed in Figure 1.2. Then, after substituting for \bar{F} , G , and H in the above equation and rearranging, we obtain

$$\dot{\hat{x}} = (A - HC)\hat{x}(t) + Bu(t) + Hy(t)$$

$$\hat{y} = C\hat{x}(t)$$

where \hat{y} is an output estimate so that the estimator can be written as follows:

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + H[y(t) - \hat{y}(t)] \quad (1.6)$$

$$\hat{y} = C\hat{x}(t) \quad (1.7)$$

where A is an $n \times n$ matrix, B is an $n \times m$ matrix, H is an $n \times p$ matrix, and C is a $p \times n$ matrix.

We impose a requirement that if $\hat{x}_0 = x_0$, then $\hat{x}(t) = x(t)$, for all $t \geq t_0$.

We denote \tilde{x} as the estimate of the difference between $x(t)$ and $\hat{x}(t)$, to be precise, $\tilde{x} \equiv \hat{x} - x$. Note, $\tilde{x}(t)$ cannot be measured as a signal since $x(t)$ is not available for measurement.

Now, we have the problem of choosing H so that $A - HC$ has a prescribed characteristic polynomial or so that the error equation (1.8) is exponentially stable. Note if all eigenvalues of the matrix $(A - HC)$ have negative real parts, then (1.8) is exponentially stable. This is a well known theorem which can be found in [4] , [13] and many textbooks on Linear Systems.

Note, generally, in mathematics this estimate error is expressed as $\delta x = \hat{x} - x$; however, in most control engineering literature, \tilde{x} is used. Also, if we restrict our attention to linear differential systems, then time-invariance is defined in terms of coefficients of the differential equations. If the coefficients of the differential equation are constant, the system is said to be time-invariant. From (1.3) with constant coefficient matrices and (1.6), $\tilde{x}(t) = \hat{x}(t) - x(t)$ satisfies the linear state equation

$$\dot{\tilde{x}}(t) = [A - HC]\tilde{x}(t) \quad (1.8)$$

$$\tilde{x}(t_0) = \hat{x}_0 - x_0$$

Therefore, (1.5) is satisfied if H can be chosen so that (1.8) properly controls the error $\tilde{x}(t)$. Such a selection of H completely specifies the linear state equations (1.6) and (1.7) that generate the estimate \hat{x} , and equation (1.6) with (1.7) is called the observer (closed loop) for the given plant. The actual value of initial estimate \hat{x}_0 is unimportant i.e. it can be taken as zero,

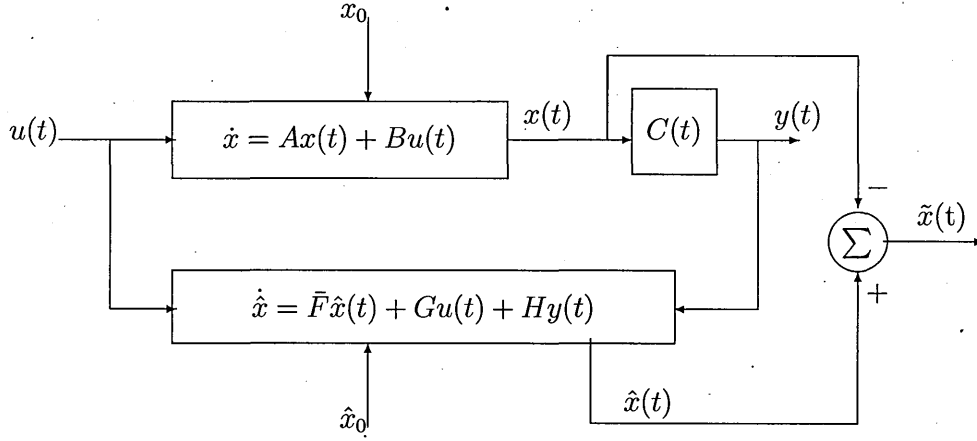


Figure 1.2: Observer structure for generating a state estimate

if we have no special information. Then the observer state $\hat{x}(t)$ can be an asymptotic state estimator by choosing an observer gain H to stabilize (1.8). That is to say, if all eigenvalues of $(A - HC)$ have negative real parts that are smaller than $-\sigma$, ($\sigma > 0$), then all the components of \tilde{x} will approach zero in the rate of $\exp(-\sigma t)$. In terms of a time-invariant case (as in this discussion), say we have $\dot{x} = Ax$, where A is a constant coefficient matrix and state variable x are of appropriate dimensions. If $\lim_{t \rightarrow \infty} e^{At} = 0$, it is said to be exponentially stable. This is a necessary and sufficient condition for exponential stability in the time-invariant case. Consequently, even if there is a large error between $\hat{x}(t_0)$ and $x(t_0)$ at initial time t_0 , the vector \hat{x} , will converge to x rapidly. This is based on a theorem in Linear Systems (see [4], Theorem 7-7), which essentially says, if a single-variable, linear time-invariant dynamical equation is observable, then an asymptotic state estimator with any set of eigenvalues can be constructed. On this subject

matter, it may be appropriate to mention a highly recommended book on *Introduction to Mathematical Control Theory* (see [23], in particular, section 4.5). Also, see [13].

1.4 Reduced-Order Observers

Section 1.3 describes the reconstruction of the state of a deterministic, control system based on exact observations of the system output. This concept of formulating an observer for reconstructing the state vector of an observable linear system from exact measurements of the output was formulated by Luenberger (see [10], [11]). A nice presentation is also given in [15].

In Section 1.5, we consider the problem of reduced-order observers in order to understand the formulation of the canonical DAE observer (see Section 2.3).

We desire to reconstruct the state of a linear dynamical system based on exact observations of the system output. Consider an n^{th} -order system described by (1.6), (1.7) (see above) which is a full-order observer with control input u . Now, let the observations of the state be available from the system according to $y(t) = Cx(t)$. Assume C , a $p \times n$ measurement matrix ($p < n$) to be full-rank. That is, $y(t)$ represents p linearly independent combinations of $x(t)$. The initial system state, x_0 , is unknown. Then a reduced-order observer that has order $(n - p)$ can be formulated which, by observing the desired system output $y(t)$, will reconstruct the state, $x(t)$, of the system exactly in an asymptotic sense. This reconstructed observer is a reduced-order observer or estimator. A major application of observer concepts has been to deterministic

feedback control problems, where the control law may depend on knowledge of all system states, while only a limited combinations of states (outputs) are measureable.

We assume that when A , B , and C are independent of t , (1.3) and (1.4) is the case of a time-invariant linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$

$$y(t) = Cx(t)$$

and, the form of the observer is (1.6), (1.7) with corresponding error state equation (1.8) as presented in Section 1.3.

Again, to reemphasize: When the estimator (observer) estimates the values of all n state variables, the problem is called the full-order state estimator problem. Thus, the observer (1.6), (1.7) is a full-order observer with corresponding error state equation (1.8).

We present an example ([13]) of a linear dynamical system where not all state variables are available or measured, to illustrate a situation in terms of the stabilization problem of system (1.3), (1.4) (with constant coefficient matrices) and, where stability cannot be achieved by output feedback. This is done to demonstrate the concept in computing an observer and a reduced-order observer.

Consider the following time-invariant state equations

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_1(t) + u(t)$$

$$y(t) = x_2(t)$$

with dimensions $n = 2$, $p = 1$, $m = 1$ and where the static linear output feedback

$$u(t) = \bar{M}y(t)$$

yields the closed-loop state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & \bar{M} \end{bmatrix} x(t)$$

The closed-loop characteristic polynomial is $\mu^2 - \bar{M}\mu - 1$ where μ are the eigenvalues. Since for any choice of \bar{M} the product of roots is -1 , one eigenvalue must be positive and hence the closed-loop state equation is not exponentially stable. This is due to the unavailability of $x_1(t)$ for use in feedback and not necessarily to the failure of observability.

Note, $x_1(t)$ and $x_2(t)$ can be used to arbitrarily assign eigenvalues. Then for the same example, we formulate the full-order observer using (1.6), (1.7) as follows

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} [y(t) - \hat{y}(t)]$$

The resulting error state equation (1.8) yields

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 - h_1 \\ 1 & -h_2 \end{bmatrix} \tilde{x}(t)$$

By setting $h_1 = 26$, $h_2 = 10$, to place both eigenvalues at -5 , we obtain exponential stability of the error state equation. Then the observer becomes

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & -25 \\ 1 & -10 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 26 \\ 10 \end{bmatrix} y(t).$$

Now to compute a reduced-order observer for the same linear state equation

in the above example, the method as given in Section 1.5 is used here. We begin with a state variable change to obtain the special form of C-matrix in (1.10). Letting

$$P = P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} \dot{z}_a(t) \\ \dot{z}_b(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0] \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix}$$

The reduced-order observer becomes the scalar state equation

$$\dot{z}_c(t) = -Hz_c(t) - Hu(t) + (1 - H^2)y(t)$$

$$\hat{z}_b(t) = z_c(t) + Hy(t)$$

For $H = 5$ we obtain the observer for $z_b(t)$ with error equation

$$\dot{\tilde{z}}_b(t) = -5\tilde{z}_b(t).$$

The observer can be written as

$$\dot{z}_c(t) = -5z_c(t) - 5u(t) - 24y(t)$$

$$\hat{z}_b(t) = z_c(t) + 5y(t)$$

$$\hat{x}(t) = \begin{bmatrix} \hat{z}_b(t) \\ y(t) \end{bmatrix}$$

Thus $\hat{z}_b(t)$ is an estimate $\hat{x}_1(t)$, while $y(t)$ provides $x_2(t)$ exactly.

Sections 1.5, 1.6 and Appendix C describe the process of obtaining a reduced-order observer. Note, for this case, we use the approach developed in Section 1.5. However, the historical approach is given in Appendix C which might be of some help to the reader in comprehending the contents of Section 1.5. The case for canonical DAE observers will be discussed in a later chapter.

1.5 Systems with Reduced-Order Observer

The purpose of this Section and the following Section 1.6 is to demonstrate the concept of Reduced-Order Observer Design of Control Systems in its original form.

In Section 1.3, the concept of observer is covered and the time-invariant estimator equation given by (1.6), (1.7) (see explanation for time-invariance following equations (1.6), (1.7)) is discussed. The requirement of a properly selected observer gain H , a constant matrix, so that $\tilde{x} = 0$ as $t \rightarrow \infty$ is as stated. The contents of this section is also used to derive equations (2.24)-(2.26) to obtain a reduced-order observer in Section 2.4.

We consider the problem of choosing another n -order linear state-equation (see Section 1.3) of the form

$$\dot{\hat{x}}(t) = \bar{F}\hat{x}(t) + Gu(t) + Hy(t), \quad \hat{x}(t_0) = \hat{x}_0 \quad (1.9)$$

with the property (1.5) and constant matrices \bar{F} , G with the same dimensions specified in (1.6), (1.7) for A , B , respectively. Thus, if $\hat{x}_0 = x_0$, then $\hat{x}(t) = x(t)$, $\forall t \geq t_0$ should hold. Now, forming the state equation for $\hat{x}(t) - x(t)$ and choosing the coefficients such that $\bar{F} = A - HC$ and $G = B$, then equation (1.9) can be written as in (1.6), (1.7), where H must be chosen so that (1.8) is satisfied.

Assume $\text{rank}(C) = p < n$, then we use a state variable transformation that leads to a reduced-order observer that has order $n - p$.

Let

$$P^{-1} = \begin{bmatrix} C \\ P_b \end{bmatrix} \quad (1.10)$$

where P_b is an $(n - p) \times n$ matrix that is arbitrary subject to the requirement that P is invertible. Then letting $z(t) = P^{-1}x(t)$, the state equation ((1.3)

and (1.4) in Section 1.3 with constant coefficient matrices) in the new state variables can be written in the partitioned form

$$\begin{bmatrix} \dot{z}_a(t) \\ \dot{z}_b(t) \end{bmatrix} = \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} \\ \bar{F}_{21} & \bar{F}_{22} \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u(t), \quad \begin{bmatrix} z_a(t_0) \\ z_b(t_0) \end{bmatrix} = P^{-1}(t_0)x_0$$

$$y(t) = [I_p \quad 0_{p \times (n-p)}] \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} \quad (1.11)$$

To see this, dropping the independent variable t for simplicity, we have the following:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Since $z = P^{-1}x$, then substitute $x = Pz$ in the above.

$$P\dot{z} = APz + Bu$$

$$\dot{z} = P^{-1}APz + P^{-1}Bu$$

$$\dot{z} = \bar{F}z + Gu$$

where $\bar{F} = P^{-1}AP$, $G = P^{-1}B$. Also, $y = CPz$.

Note, \bar{F}_{11} is $p \times p$, G_1 is $p \times m$, $z_a(t)$ is $p \times 1$, and the remaining partitions have corresponding dimensions. We also have, $z_a(t) = y(t)$.

Now we want to show how to obtain the asymptotic estimate of $z_b(t)$ needed to obtain an asymptotic estimate of $x(t)$.

Assume that we have computed an $(n-p)$ order-observer for $z_b(t)$ that has a form different from the full-order case as follows:

$$\dot{z}_c(t) = \tilde{F}z_c(t) + \tilde{G}_a u(t) + \tilde{G}_b z_a(t)$$

$$\hat{z}_b(t) = z_c(t) + Hz_a(t) \quad (1.12)$$

For known $u(t)$ and for any initial values $z_b(t_0)$, $z_c(t_0)$, $z_a(t_0)$ and the resulting $z_a(t)$ from (1.11), the solutions of (1.11) and (1.12) are such that

$$\lim_{t \rightarrow \infty} [\hat{z}_b(t) - z_b(t)] = 0$$

Then an asymptotic estimate for the state vector in (1.11), where the first p components are exact, can be written as

$$\begin{bmatrix} \hat{z}_a(t) \\ \hat{z}_b(t) \end{bmatrix} = \begin{bmatrix} I_p & O_{p \times (n-p)} \\ H & I_{n-p} \end{bmatrix} \begin{bmatrix} y(t) \\ z_c(t) \end{bmatrix}$$

Adopting this variable-change set up, we then examine the problem of computing $(n-p)$ order observer of the form (1.12) for an n^{th} - order state equation in the special form (1.11). Then, we need to show that the $(n-p) \times 1$ error signal

$$\tilde{z}_b(t) = \hat{z}_b(t) - z_b(t) \quad (1.13)$$

satisfies the error state expression for $\dot{\tilde{z}}_b(t)$. Using the equations from (1.11) and (1.12), $\dot{z}_b(t)$, $\dot{\hat{z}}_b(t)$, and $\dot{z}_c(t)$ are substituted in (1.13). After rearranging, we obtain the following equation:

$$\begin{aligned} \dot{\tilde{z}}_b(t) &= \tilde{F}\tilde{z}_b(t) + [\bar{F}_{22} - H\bar{F}_{12} - \tilde{F}]z_b(t) \\ &+ [\bar{F}_{21} + \tilde{F}H - \tilde{G}_b - H\bar{F}_{11}]z_a(t) + [G_2 - \tilde{G}_a - HG_1]u(t) \end{aligned}$$

and the initial condition $\tilde{z}_b(t_0) = \hat{z}_b(t_0) - z_b(t_0)$. Then, regardless of $u(t)$, $z_a(t_0)$ and the resulting $z_a(t)$, and since $z_b(t_0) = \hat{z}_b(t_0)$ should yield $\tilde{z}_b(t) = 0$, $\forall t \geq t_0$, we make the coefficients as follows:

$$\tilde{F} = \bar{F}_{22} - H\bar{F}_{12}$$

$$\tilde{G}_b = \bar{F}_{21} + \tilde{F}H - H\bar{F}_{11}$$

$$\tilde{G}_a = G_2 - HG_1$$

with the resulting $(n - p) \times 1$ error state equation

$$\dot{\tilde{z}}_b(t) = [\bar{F}_{22} - H\bar{F}_{12}]\tilde{z}_b(t), \quad \tilde{z}_b(t_0) = \hat{z}_b(t_0) - z_b(t_0)$$

For the reduced-order observer in (1.12), the $(n - p) \times p$ gain H should satisfy the stability criteria (see (1.5)).

In terms of the original state equation (1.6), (1.7) leads to the asymptotic estimate for $x(t)$ via

$$\hat{x}(t) = P \begin{bmatrix} I_p & O_{p \times (n-p)} \\ H & I_{n-p} \end{bmatrix} \begin{bmatrix} y(t) \\ z_c(t) \end{bmatrix} = P \begin{bmatrix} y(t) \\ \hat{z}_b(t) \end{bmatrix}$$

Then, $n \times 1$ estimate error $\tilde{x}(t) = \hat{x}(t) - x(t)$ is given by

$$P[\hat{z}(t) - z(t)] = P \begin{bmatrix} 0 \\ \tilde{z}_b(t) \end{bmatrix}$$

(recall that $y(t) = z_a(t)$, where p components are exact, then $\tilde{z}_a(t) = 0$).

A reduced-order observer is computed using the discussion above for the time-invariant linear state equation in the example presented in Section 1.6.

Summarizing: we achieved a way to use y (with dimension less than n) to create an estimate $\hat{x}(t)$ of $x(t)$ that is asymptotically exact.

1.6 Example

An example from [7] is presented for the purpose of demonstrating the adaptation of the previous sections on reduced-order observer. A second order simplified tracking system problem is considered, where x_1 and x_2 are position and velocity of the tracked object, respectively. u is the input which

drives the system. The differential equations representing the dynamics of the system is as follows:

$$\dot{x} = Ax + Bu \quad (1.14)$$

with measurements of position available according to

$$y = Cx \quad (1.15)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (1.16)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \quad (1.17)$$

$$B = \begin{bmatrix} 0 \\ l \end{bmatrix} \quad (1.18)$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (1.19)$$

It is desired to construct an observer for this system. Refer to Figure 1.3 to see the system model representing this tracking system. The above can also be expressed as a system of linear differential equations as follows:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\beta x_2 + lu$$

together with

$$y = x_1$$

Now, from the methodology as developed in the previous section a reduced-order observer is obtained as follows:

A, B, C are as given in the example. Letting $P^{-1} = P = I$ yields (1.11), where $\bar{F}_{11} = 0, \bar{F}_{12} = 1, \bar{F}_{21} = 0, \bar{F}_{22} = -\beta, G_1 = 0, G_2 = l$, and

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} z(t), \quad z(t)^t = \begin{bmatrix} z_a(t) & z_b(t) \end{bmatrix}.$$

Then

$$\dot{z}_c = -(\beta + H)z_c - (\beta + \beta H)Hy + lu$$

and we obtain an observer for $z_b(t)$ with error equation

$$\dot{\tilde{z}}_b = -(\beta + H)\tilde{z}_b$$

and for $\beta = 1$ and $H = 4$

$$\dot{z}_c = -5z_c - 20y + lu$$

$$\hat{z}_b = z_c + 4y$$

If we equate $y = \hat{x}_1, \hat{z}_b = \hat{x}_2$ and $z_c = \hat{\xi}$, we can compare with the results as obtained in Appendix C. That is, the two equations above yield

$$\dot{\hat{\xi}} = -5\hat{\xi} - 20\hat{x}_1 + lu$$

$$\hat{x}_2 = \hat{\xi} + 4\hat{x}_1$$

Note, the block diagram as illustrated in Figure 1.4 corresponds to

$$\hat{x}_1(t) = y(t)$$

$$\hat{x}_2(t) = \hat{\xi}(t) + 4\beta\hat{x}_1(t)$$

as derived in Appendix C.

The construction of Figure 1.4 is based on the following equations from Appendix C:

$$\dot{\hat{\xi}}(t) = -5\hat{\xi}(t) - 5\beta by + lu$$

$$\hat{x}_2 = \hat{\xi} + 4\beta\hat{x}_1$$

The observer performance can be illustrated by a numerical example. Assume that the system parameters are $\beta = 1.0 \text{ sec}^{-1}$, $l = 1.0$, $u(t) = -1.0 \text{ ft/sec}^2$, with $x_1(0) = 1.0 \text{ ft}$, and $x_2(0) = 1.0 \text{ ft/sec}$ given. The observer is initialized by choosing $\hat{x}_1(0) = 1$, $\hat{x}_2(0) = 0$, then from (1.9) we have $\hat{\xi}(0) = -4.0 \text{ ft/sec}$. Using these initializations, and following Figures 1.3, 1.4, we can compute the values $[x_1(t), x_2(t)]$, $[\hat{x}_1(t), \hat{x}_2(t)]$ and compare the results between the outputs of the system and observer.

Note, Figures 1.3, 1.4 are state variables diagrams using state-space representations of the dynamical systems as given by their differential equations, respectively, for the tracking system and for its observer design. This diagram form is very useful in connection with system modeling and is commonly used in modern control theory to assist one in solving differential systems, either by analytical means or by numerical techniques using a computer. Often the diagrams give greater engineering insight from the model. The diagram is very close to hardware layout for electronic implementation. From a theoretical perspective such a diagram sometimes reveals structural features of the linear state equation that are not apparent from the coefficient matrices.

Given the parameters and initial conditions, the solution to the differential equation system (1.14) - (1.19) is as follows:

$$x_1(t) = 3 - 2e^{-t} - t$$

$$x_2(t) = 2e^{-t} - 1$$

Because $\hat{x}_1(t) = x_1(t)$,

$$\hat{x}_1(t) = 3 - 2e^{-t} - t$$

Then,

$$\hat{x}_2(t) = \hat{\xi}(t) + 4(3 - 2e^{-t} - t) = 12 - 8e^{-t} - 4t + \hat{\xi}(t)$$

$\hat{\xi}(t)$ is obtained by solving the differential equation

$$\dot{\hat{\xi}} + 5\hat{\xi}(t) = -20(3 - 2e^{-t} - t) - 1$$

using either the integrating factor e^{5t} and integrating, or by Laplace Transform as shown in Appendix B. Solving for $\hat{\xi}(t)$ above, yields the following:

$$\hat{x}_2(t) = -1 + 2e^{-t} - e^{-5t}$$

The graphs for $x_1(t)$, $\hat{x}_1(t)$ and $x_2(t)$, $\hat{x}_2(t)$ in Figures 1.5, 1.6, respectively, were plotted using MATLAB software.

Now, comparing $x = (x_1, x_2)$ with $\hat{x} = (\hat{x}_1, \hat{x}_2)$, we have that x_1 is estimated without error since it is directly measured, while the error in the estimate of x_2 decays exponentially to zero. This is seen in the following way: since $\tilde{x}_2 = \hat{x}_2 - x_2 = -e^{-5t}$ then $\lim_{t \rightarrow \infty} |\hat{x}_2(t) - x_2(t)| \rightarrow 0$, but at $t = 0$, $|\tilde{x}_2| = 1$, and at $t = 1.0$, $|\tilde{x}_2| = .006$. The difference between x_2 and \hat{x}_2 approaches zero more rapidly than e^{-t} as $t \rightarrow \infty$ ($t > 1$).

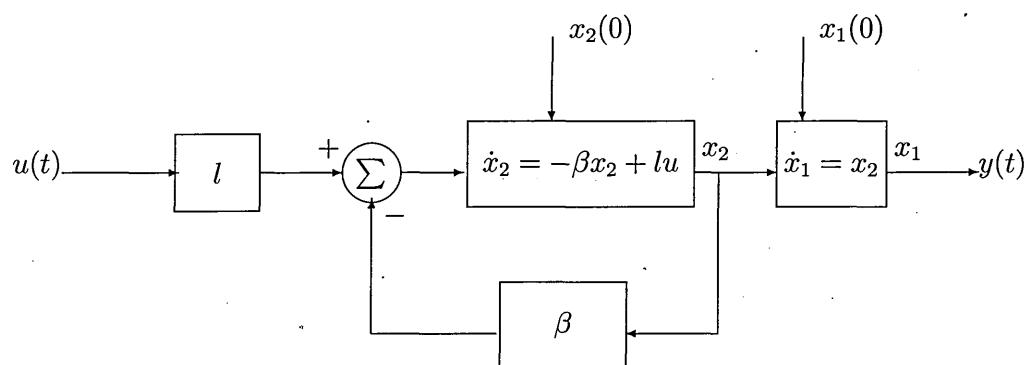


Figure 1.3: System model of a 1-dimensional tracking problem where x_1 and x_2 are position and velocity of tracked object, respectively.

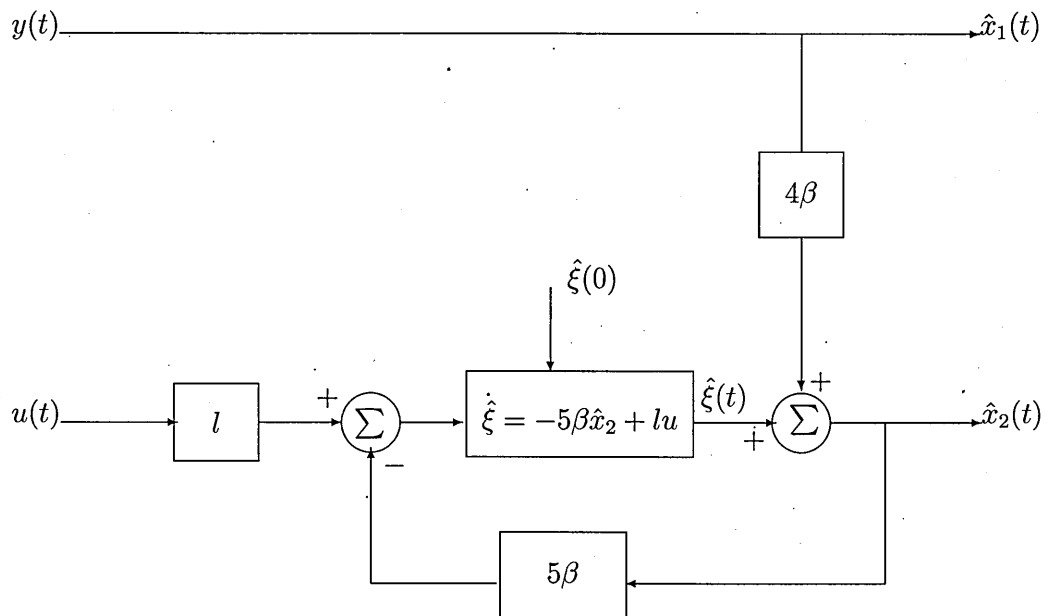


Figure 1.4: Observer design of tracking example in Figure 1.3 using transformation as described in Section 1.6

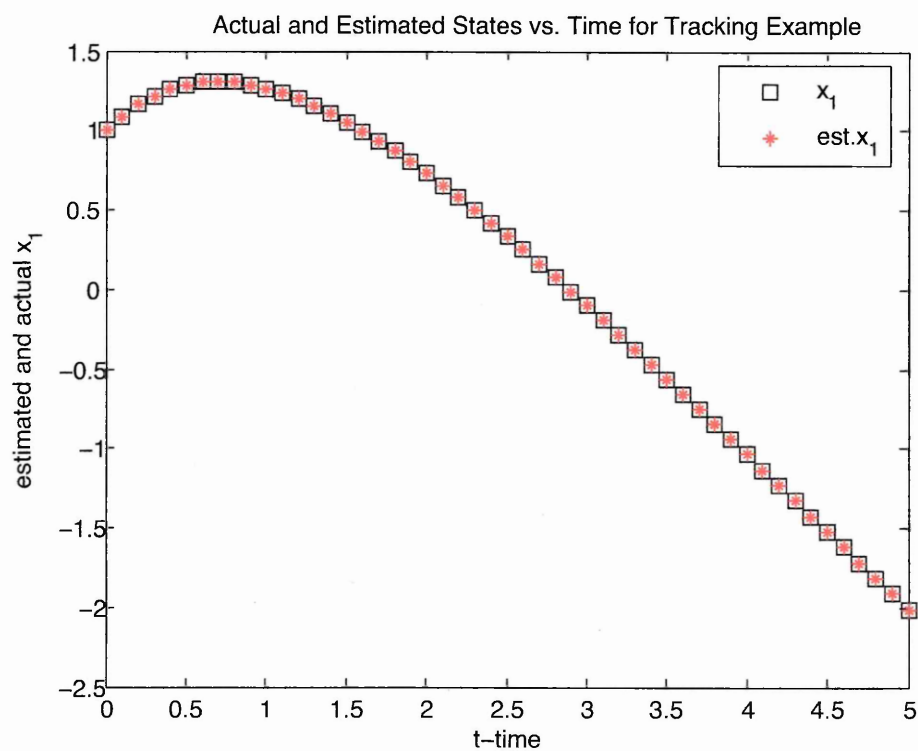


Figure 1.5: Graph of state variables $x_1(t)$ (actual) and $\hat{x}_1(t)$ (estimated) in

tracking example. $\hat{x}_1(t)$ estimates $x_1(t)$ exactly for time t . That is, $\hat{x}_1(t) =$

$$x_1(t) = 3 - 2e^{-t} - t$$

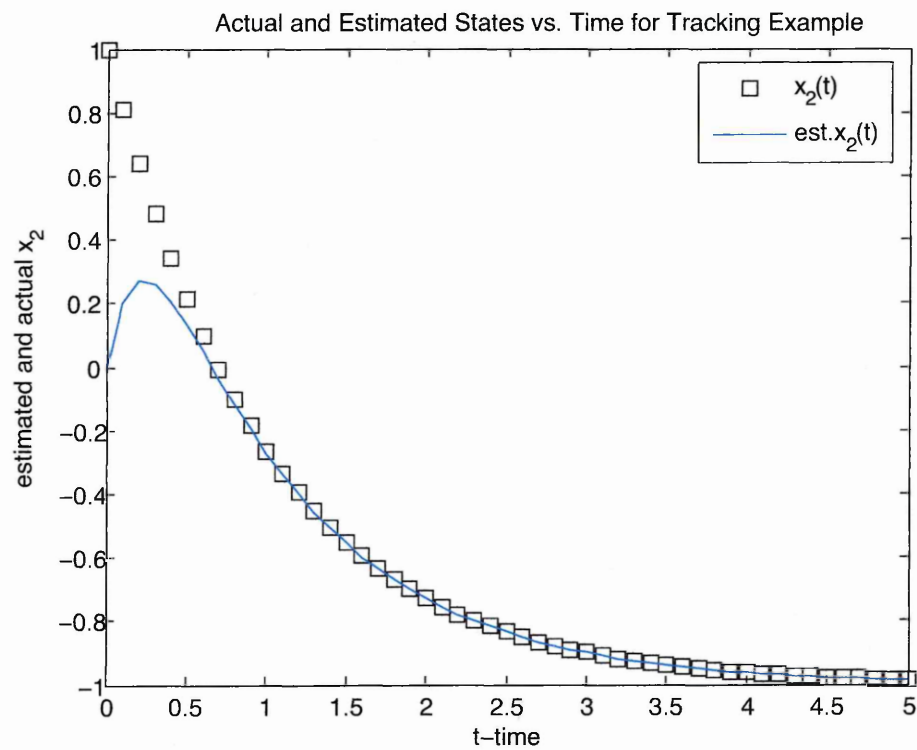


Figure 1.6: Graph of state variables $x_2(t)$ (actual) and $\hat{x}_2(t)$ (estimated) in

tracking example. $\hat{x}_2(t)$ estimates $x_2(t)$ closely for time t , where $\hat{x}_2(t) =$

$$-1 + 2e^{-t} - e^{-5t} \text{ and } x_2(t) = -1 + 2e^{-t}$$

Chapter 2

Formulation of DAE Observer

2.1 DAE Observer System

In this chapter, we present the descriptor formulation (see Section 2.3) of the observer as an implicit system. In this form, the observer provides an extra degree of freedom which is useful in many of the observer design methods described in [5] and [9], and may be solved using a DAE solver. In [3], it is shown that a DAE solver can compute the solution of the algebraic equations by Newton iterations at every time step without requiring the development of own-code by the implementor.

Note that differential equation system (1.1), (1.2) is a DAE system in $x \in \mathbb{R}^n$

(unknown or measured), $u \in \mathbb{R}^m$ ($m \leq n$ known control input), and $y \in \mathbb{R}^p$ (known output), as given in Section 1.2; hence we have n unknowns, and $n + p$ equations where $p \leq n$. Then this DAE is overdetermined and it describes all constraints (information) that we have for constructing estimate \hat{x} . Equation $y = h(x)$ is normally used to compute y from the measured x ; otherwise, such computation will be subject to errors due to model uncertainty. In his paper [12], Nikoukhah basically describes the construction of \hat{x} , by means of introducing a relaxation variable $\lambda = \lambda(t)$, which will make this DAE integrable and yield the usual explicit formulation of the observer. This concept is covered in Sections 2.2 and 2.3.

In order to discuss the reformulation of an ODE to a DAE observer, the definition of an “index” of a DAE and the utilization of its concept are introduced in the next section. The evaluation of the “index” is shown to determine the order of complexity in generating a solution or solutions to a DAE observer. Its relationship with reduced-order observers, a topic covered in previous Sections 1.4 and 1.5, is considered as well.

2.2 Index of DAE

Note: The notation in this Section should not be confused with that of previous Sections. It is self-contained for the purpose of stating definitions needed for later Sections.

Consider the first order system $F(t, y, \dot{y}) = 0$ (given initial condition y_0 at t_0) where $y(t)$ is a function belonging to an open set $Y \subset \mathbb{R}^n$ and F is vector

valued. The index of a DAE satisfying initial conditions is the minimum number of differentiations of the system which would be required to solve for \dot{y} uniquely in terms of y and t (i.e. to define an ODE for y). If $\frac{\partial F}{\partial \dot{y}}$ is nonsingular for all t then \dot{y} can be determined in terms of y and t , the index is zero, and this is an implicit ODE i.e. no differentiation is necessary. In general, $F(t, y, \dot{y})$ is differentiated as many times as may be needed to solve for \dot{y} uniquely in terms of y and t . Thus, the index is defined in terms of the system

$$\begin{aligned} F(t, y, \dot{y}) &= 0 \\ \underline{G}_1(t, y, \dot{y}, \ddot{y}) &= \frac{dF}{dt}(t, y, \dot{y}) = 0 \\ &\vdots \\ \underline{G}_l(t, y, \dots, y^{(l+1)}) &= \frac{d^l F}{dt^l}(t, y, \dot{y}) = 0 \end{aligned} \tag{2.1}$$

to be the smallest integer l ($l < n$) so that $\dot{y}, \ddot{y}, \dots, y^{(l+1)}$ in the above can be solved for in terms of y and t using initial conditions. Differentiating $F(t, y, \dot{y})$ with respect to t yields $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} \dot{y} + \frac{\partial F}{\partial \dot{y}} \ddot{y} = 0$. Note, the differentiation of (2.1) is rarely done in computation. However, such a definition helps us to understand the underlying structure of the DAE system.

Roland England et al [16], in a paper mainly concerned with the transformation of a general optimal control problem to a system of DAEs, presents a technique for determining the index of the problem and introduces the Tractability Index which is a useful tool for determining the index of a general system of DAEs. Examples of DAEs of index 1-3 are presented and the solutions to differential equations are performed analytically. Appendix A contains a simple example demonstrating the technique described in [16] for representing an optimal control problem as a DAE and determining its solution.

A DAE system of the form

$$\begin{aligned}\dot{x} &= f(t, x, z) \\ 0 &= g(t, x, z)\end{aligned}$$

where f, g are of \mathcal{C}^2 (derivatives being with respect to t), $x \in \mathbb{R}^n$, and $z \in \mathbb{R}^p$ is a semi-explicit DAE or an ODE with constraints which is a special case of $F(t, y, \dot{y}) = 0$ and is of index 1 provided $\partial g / \partial z$ is nonsingular (z being the same dimension as g). For semi-explicit index-1 DAE, we can distinguish between differential variables $x(t)$ and the algebraic variable $z(t)$. Only one differentiation of $g(t, x, z)$ with respect to t yields \dot{z} and in principle using the implicit function theorem one can solve the new system for z . To see this, we differentiate $g(t, x, z)$ with respect to t to obtain, $\dot{z} = -(g_z)^{-1}(g_t + g_x f)$, where g_t is a vector, and g_x, g_z are Jacobian matrices.

Alternatively, substituting $z = \bar{g}(t, x)$ (obtained from solving $g(t, x, z) = 0$ for z) in $\dot{x} = f(t, x, z)$ would produce an ODE in x only, which is possible if $\partial g / \partial z$ is nonsingular over the trajectory defined by the equations.

The Hessenberg Index-1 form is also of this type, i.e. semi-explicit index-1 DAE having the additional property that $g_x = 0$.

Another type of a DAE system is the following

$$\begin{aligned}\dot{x} &= f(t, x, z) \\ 0 &= \bar{g}(t, x)\end{aligned}$$

where the constraint \bar{g} is devoid of z . This is a pure index-2 DAE and all the algebraic variables play the role of index-2 variables. This type is sometimes referred to as Hessenberg Index-2. To see this, we differentiate the constraint with respect to t and obtain $0 = \bar{g}_t + \bar{g}_x \dot{x} = \bar{g}_t + \bar{g}_x f$ and differentiating again

with respect to t , $0 = \bar{g}_{tt} + 2\bar{g}_{xt}f + \bar{g}_x f_t + \bar{g}_x f_z \dot{z} + \bar{g}_x f_x f + \bar{g}_{xx} f^2$. Since we have twice performed differentiation, the index is 2 provided $\bar{g}_x f_z$ is nonsingular for all t . Rewriting the expression above, we have

$$\dot{z} = -[\bar{g}_x f_z]^{-1}(\bar{g}_{tt} + 2\bar{g}_{xt}f + \bar{g}_x f_t + \bar{g}_x f_x f + \bar{g}_{xx} f^2).$$

In the case of canonical DAE observer, discussed below, higher index than one is not considered. However, higher-index DAEs are discussed in [14]. Special DAE forms such as Hessenberg index 1-3 are covered, where examples are given demonstrating how higher-index DAEs are expressed as a combination of more restrictive structures of ODEs coupled with constraints while identifying algebraic and differential variables. Some or all of the algebraic variables may be eliminated using the same number of differentiations.

The following is a discussion of a theorem and definitions adapted from a different source [3].

Definitions:

The nonlinear DAE $F(t, y, \dot{y}) = 0$ is said to be uniform index one if the index of the constant coefficient system

$$U\dot{\bar{w}}(t) + V\bar{w}(t) = g(t)$$

where $U = F_{\dot{y}}(t, y, \dot{y})$, $V = F_y(t, y, \dot{y})$ and $g(t) = -F_t(t, y, \dot{y})$, is one for all (t, y, \dot{y}) in the neighborhood of the graph of solution of $F(t, y, \dot{y})$, and if

- 1) The partial derivatives of U with respect to t, y, \dot{y} exist and are bounded in a neighborhood of the solution
- 2) The rank of U is constant in a neighborhood of the solution.

The index of $F(t, y, \dot{y})$ is determined by computing the determinant of the matrix pencil, $\{U, V\} \equiv \bar{\rho}U + V$, where $\bar{\rho}$ is a complex parameter.

If $\det(\bar{\rho}U + V)$ is not identically zero as a function of $\bar{\rho}$ then the pencil is said to be regular, or a regular pencil.

◇

By Theorem 2.3.2 in [3], if $\bar{\rho}U + V$ is a regular pencil, then there exist nonsingular matrices \bar{P}, \bar{Q} such that

$$\bar{P}U\bar{Q} = \begin{bmatrix} I & 0 \\ 0 & \bar{N} \end{bmatrix}, \quad \bar{P}V\bar{Q} = \begin{bmatrix} \bar{C} & 0 \\ 0 & I \end{bmatrix}$$

where \bar{N} is a matrix of nilpotency l (defined as the index of $\{U, V\}$ which is also the index of the DAE) and I is an identity matrix. This form is important when considering the matrix $(U + \bar{\rho}V)^{-1}$ in numerical methods for solving DAE; however, this is outside of the scope of this thesis. When $\bar{N} = 0$, $l = 1$ is the degree of nilpotency. A zero matrix has nilpotency $l = 1$ since $\bar{N} = 0$.

The index of our DAE $F(t, y, \dot{y})$ is one, since only one differentiation of the algebraic constraint with respect to t is required. The relationship of the definitions for index relative to canonical DAE observer will be presented in the next section and a later section on Numerical Implementation.

2.3 Canonical DAE Observer

Consider the multivariable nonlinear system as described by (1.1), (1.2) in Sections 1.2 and 2.1. This DAE system is overdetermined and describes all the constraints that we have for constructing a state variable, \hat{x} (as mentioned previously), and our goal is to make this integrable. First, a relaxation variable $\lambda(t)$ of dimension p is introduced into this DAE in such a way that

the resulting DAE has index one. That is, let $\lambda \in \mathbb{R}^p$ be used in an explicit formulation of the observer, where $w(\lambda, \dot{\lambda})$ contains the relaxation variable λ and its derivative $\dot{\lambda}$. The unknowns are \hat{x} and λ with y explicitly known as a function of \hat{x} and u is the input as in the previous chapter. Then, we have the integrable form as follows:

$$\dot{\hat{x}} = f(\hat{x}, u) + w(\lambda, \dot{\lambda}) = \bar{f}(\hat{x}, u, \lambda, t) \quad (2.2)$$

$$0 = y - h(\hat{x}) = \underline{g}(\hat{x}, y) \quad (2.3)$$

By letting

$$w(\lambda, \dot{\lambda}) = h_x(\hat{x})^t \dot{\lambda} + v(\hat{x}, u, \lambda, t) \quad (2.4)$$

where v is any function of \hat{x} , u , λ , and t (time), we formulate a DAE observer such that its index does not exceed 1 as shown below (viz. lemma 1 in [12]). Note $h_x(\hat{x})^t$ is the transpose of $h_x(\hat{x})$ and differentiating $y = h(\hat{x})$ with respect t , yields $\dot{y} = h_x(\hat{x})\dot{\hat{x}}$. Replacing $w(\lambda, \dot{\lambda})$ in (2.2) with (2.4), we formulate the following matrix differential equation system:

$$\begin{bmatrix} I & -h_x(\hat{x})^t \\ -h_x(\hat{x}) & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} f(\hat{x}, u) + v(\hat{x}, u, \lambda, t) \\ -\dot{y} \end{bmatrix} \quad (2.5)$$

The above equation is in fact an ODE because $h_x(\hat{x})$ has a full row rank by assumption (see Section (1.2)) and the matrix on the left-hand side is nonsingular and, thus, invertible, since the symmetric matrix $(h_x(\hat{x})h_x(\hat{x})^t)$ is positive definite. Thus, we have a DAE in a form whose index does not exceed 1. Note, nonsingular ODEs have index 0.

Consider, the following example to emphasize the above:

Let the ODE system (2.5) be a nonlinear differential system of order three, where the left-hand matrix coefficient is designated D ,

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad h_x = \begin{bmatrix} h_{x_1} \\ h_{x_2} \end{bmatrix}, \quad \hat{x} \in \mathbb{R}^2, \quad y \in \mathbb{R}, \quad u \in \mathbb{R}, \quad \lambda \in \mathbb{R}$$

Then D is a 3×3 matrix i.e.

$$D = \begin{bmatrix} I & -h_x(\hat{x})^t \\ -h_x(\hat{x}) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -h_{x_1} \\ 0 & 1 & -h_{x_2} \\ -h_{x_1} & -h_{x_2} & 0 \end{bmatrix}$$

Since the $\text{rank}(D) = 3$ (determined by elementary row operations), D is nonsingular and invertible. That is, $|D| = -h_{x_1}^2(\hat{x}) - h_{x_2}^2(\hat{x}) < 0$. Note, no more than one differentiation of the algebraic equation is needed ($\dot{y} = h_x(\hat{x})\dot{\hat{x}}$). Then this DAE system,

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\lambda} \end{bmatrix} = -D^{-1} \begin{bmatrix} f(\hat{x}, u) + v(\hat{x}, u, \lambda, t) \\ -\dot{y} \end{bmatrix}$$

is index zero. and DAE system (2.2), (2.3) is of index 1.

Section 2.2 contains a brief statement to limit this thesis to canonical DAE observer of index 1 as DAEs of higher index are beyond the scope of this thesis.

The formulation of the following Canonical DAE Observer [12]

$$\dot{\hat{x}} = f(\hat{x}, u) + h_x(\hat{x})^t \dot{\lambda} + \Gamma(\hat{x}, u)\lambda \quad (2.6)$$

$$0 = y - h(\hat{x}) \quad (2.7)$$

is based upon the necessity of (2.2), (2.3) and (2.4) being the proper relaxation of (1.1) and (1.2). That is, (2.2), (2.3) and (2.4) is a proper relaxation of (1.1), (1.2), if by setting $\lambda = 0$ in (2.2), (2.3) and (2.4), we obtain the

equations in (1.1), (1.2). Thus, we must have $v(\hat{x}, u, 0, t) = 0$. Note, y and λ have the same dimension p .

Here, we consider the case $v(\hat{x}, u, \lambda, t) = \Gamma(\hat{x}, u)\lambda$ despite the fact that this is not the most general formulation. This approach is useful in obtaining far-reaching results in a later chapter when its application to interconnected subsystems is covered.

Although (2.6), (2.7) is a proper relaxation and has index 1 (as seen from (2.5)), this does not guarantee that the variables $\lambda, \dot{\lambda}$ converge to zero, which under the observability assumption (see Section 1.3) implies that \hat{x} converges to the unknown state, x , for the proper choice of Γ . Note, if $y = h(\hat{x}) + \bar{k}u$ where \bar{k} is a constant matrix then $\bar{k}_x = 0$ and $\bar{h}_x(\hat{x}, u) = h_x(\hat{x})$, where $\bar{h}(\hat{x}, u)$ is in affine form (see Section 1.2).

The resemblance between the formulation of the canonical DAE observer and a standard observer is seen in that they both are state estimators where that part of the state which is directly observed is the same. However, in system (1.6), (1.7), we are concerned with appropriately choosing the observer gain H to stabilize (1.8) as described in Section 1.3, but in system (2.6), (2.7), λ and Γ must be appropriately selected in order to guarantee stability of DAE error estimation.

2.4 Convergence Issue

This section considers the convergence issue by examining an example as in [12].

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= \sin(x_2) \\ \dot{x}_2 &= -x_2 + x_1 u \\ y &= x_2 \end{aligned} \tag{2.8}$$

where input $u(t)$ is uniformly continuous and bounded and $x_1(t)$ and $x_2(t)$ are not known. Note, y and λ have dimension one.

The Canonical DAE Observer becomes as shown below:

$$\begin{aligned} \dot{\hat{x}}_1 &= \sin(\hat{x}_2) + \gamma_1 \lambda + h_{x_1}(\hat{x}) \dot{\lambda} \\ \dot{\hat{x}}_2 &= -\hat{x}_2 + \hat{x}_1 u + \gamma_2 \lambda + h_{x_2}(\hat{x}) \dot{\lambda} \\ 0 &= y - \hat{x}_2 \end{aligned} \tag{2.9}$$

but $\Gamma = [\gamma_1, \gamma_2]^t$ and $h = x_2$, $h_x(\hat{x}) = [0, 1]^t$; then for this problem it follows that

$$\begin{aligned} \dot{\hat{x}}_1 &= \sin(\hat{x}_2) + \gamma_1 \lambda \\ \dot{\hat{x}}_2 &= -\hat{x}_2 + \hat{x}_1 u + \gamma_2 \lambda + \dot{\lambda} \\ 0 &= y - \hat{x}_2 \end{aligned} \tag{2.10}$$

and

$$\begin{bmatrix} \sin(\hat{x}_2) \\ u\hat{x}_1 - \hat{x}_2 \end{bmatrix} = f(\hat{x}, u) \tag{2.11}$$

as required in (2.6), (2.7). Differentiating y with respect to t , we obtain $\dot{y} = h_x(\hat{x})\dot{\hat{x}} = h_x(\hat{x})(f(\hat{x}, u) + h_x(\hat{x})^t \dot{\lambda} + \Gamma \lambda)$. Then, using (2.5) yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \sin(\hat{x}_2) + \gamma_1 \lambda \\ u\hat{x}_1 - \hat{x}_2 + \gamma_2 \lambda \\ -\dot{y} \end{bmatrix}$$

We have a DAE (actually, an ODE of index 0). We note that (2.10) is a DAE of index 1 in \hat{x} and λ , since the last equation of (2.10) need only be differentiated once. We also observe that this DAE is integrable using standard DAE solvers.

For the convergence of \hat{x} to the unknown state x , we perform an error analysis by using the error equation, $\tilde{x} = \hat{x} - x$, i.e. by subtracting DAE (2.10) from the system equation (2.8), where $\tilde{x}_2 = \hat{x}_2 - x_2 = 0$ and $\sin(\hat{x}_2) - \sin(x_2) = 0$ since \hat{x}_2 is exact, the following is obtained:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \gamma_1 \lambda \\ 0 &= \tilde{x}_1 u + \dot{\lambda} + \gamma_2 \lambda \end{aligned} \tag{2.12}$$

Then the error equation for (2.12) is expressed as a matrix in the following way :

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 & \gamma_1 \\ -u & -\gamma_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \lambda \end{bmatrix} \tag{2.13}$$

The above is a linear error equation and depends on input u which is measured and thus known.

The next objective is to find a $\Gamma = [\gamma_1, \gamma_2]$ such that $\tilde{x}_1, \lambda \rightarrow 0$ is guaranteed i.e. $[\tilde{x}_1, \lambda] = [0, 0]^t$ is the equilibrium point for (2.13). This error system suggests the application of the Lyapunov Stability Theorem (Second Method of Lyapunov) whose advantage is in determining the stability of the system without solving the differential equation.

This theorem can be found in many textbooks on Linear and/or Nonlinear Systems, e.g. [9], [13]. It basically says that, if a positive-definite function $\bar{V}(\tilde{x}, t)$ can be found such that $\dot{\bar{V}}(\tilde{x}, t)$ is a negative-definite function, then the origin is asymptotically stable. Based on this theorem, we can choose the following energy equation (Lyapunov function):

$$\bar{V}(\tilde{x}_1, \lambda) = \tilde{x}_1^2 + \bar{\beta}\lambda^2 \text{ (positive definite)}$$

where constant $\bar{\beta} > 0$. Differentiating with respect to t , we obtain

$$\dot{\bar{V}} = 2\tilde{x}_1\dot{\tilde{x}}_1 + 2\bar{\beta}\lambda\dot{\lambda}$$

then after substituting for $\dot{\lambda}$, $\dot{\tilde{x}}_1$

$$\dot{\bar{V}} = 2(\gamma_1 - \bar{\beta}u)\tilde{x}_1\lambda - 2\bar{\beta}\gamma_2\lambda^2.$$

If $\gamma_1 = \bar{\beta}u$, $\gamma_1 u > 0$, and $\gamma_2 > 0$ so that \bar{V} is non-increasing, $\dot{\bar{V}}$ is negative semi-definite, and $\lambda \rightarrow 0$. The system is also asymptotically stable based on Lyapunov's theorem. $\dot{\lambda}$ also converges to zero because of uniform continuity. Thus λ is a proper relaxation. (Recall the discussion following equations (2.6), (2.7) in the previous Section concerning proper relaxation for a proper choice of Γ). In addition, u must not be zero; otherwise, x_1 would be decoupled from y and x_1 could not be estimated. Because of the second equation in (2.12) and $\dot{\lambda}, \lambda \rightarrow 0$, we have that $\tilde{x}_1 \rightarrow 0$ i.e. $\hat{x}_1 \rightarrow x_1$. Thus, in order to guarantee observability, $u \neq 0$ must be persistently active as $\lambda, \dot{\lambda} \rightarrow 0$. That is to say, if u is constant and zero, x_1 is decoupled from y and cannot be estimated. Then the system is not observable for all time, t . That is, no response or output can be observed.

Information concerning stability of the origin can be obtained by computing the eigenvalues of the matrix in (2.13), which are $\frac{1}{2}(-\gamma_2 \pm \sqrt{\gamma_2^2 - 4\gamma_1 u})$. If

$\gamma_2 > 0$ and $\gamma_2^2 > 4\gamma_1 u$, then $\tilde{x}_1 \rightarrow 0$ and $\lambda \rightarrow 0$. However, by using this theorem not only do we have information about system stability but also about λ , when a Lyapunov function is used. It should be noted that this theorem is more general since it works even if u is time-varying; thus, the use of the Lyapunov function is the preferred technique. In Chapter 4, this method will be applied again.

In the above example, the canonical DAE observer is linear; however, in general it may not be linear for standard ODE observers. In the DAE approach, substituting a measured state common to the original system and the observer, may lead to a simpler “linear error” equation compared to the original system or what would be obtained using standard observer. This is a major advantage of the DAE approach used in this example and shall be applied to interconnected subsystems.

2.5 Sensitivity to Noise

From the last section, the index of descriptor formula (2.2),(2.3) was found to be one, provided $\underline{g}_y(\hat{x}, y)$ is nonsingular (see Section 2.2); otherwise, the index may be two or greater depending on the number of times required to differentiate the algebraic equations to obtain an ODE. However, high-frequency noise is of a major concern and needs to be addressed since differentiation of algebraic equations can cause the solution to become very sensitive to noise as in the problem of reduced-order observers (illustrated later in this Section). This is claimed in [4], as well as in [8]. The canonical DAE observer of index one formulated in (2.6), (2.7) will be the main focus in the follow-

ing sections. However, this formulation is closely related to reduced-order observers in the sense that part of the state which is directly observed is estimated immediately, as was shown in Sections 1.4, 1.5 and 1.6. DAEs of higher than index one may be a problem in its sensitivity to noise; and, by dealing with DAE of index one, solutions with sensitivity to noise is kept to a minimum or may even be devoid of noise sensitivity. This allows the estimated solution $\hat{x}(t)$ to be computed with greater accuracy.

Although there exist powerful DAE solvers for index two and three systems with special structures, the only case where DAEs solvers do not require that the system have any special structure and work as reliably as ODE solvers do is index one (see [12]). The DAE given by (2.2), (2.3) being of index one is what makes it tractable where differentiating (2.3) yields an ODE in \hat{x} and λ which is not of any special form. Note, (2.5) is a ODE of index zero.

This DAE formulation affords an approach which is applicable to a very special case to be covered in a later section on interconnected control subsystem. This technique will be used to solve multi-input feedback system.

It is shown by an example in [12], that canonical DAE observer does not possess the noise sensitivity that may be present in reduced-order observers by comparing the results from their solutions. In this section, this example will be developed so that the variation in the sensitivity to noise can be demonstrated.

Consider the following linear system

$$\dot{x}_1 = x_2 \tag{2.14}$$

$$\dot{x}_2 = 0 \tag{2.15}$$

$$y = x_1 \tag{2.16}$$

In the following sub-sections, the plan is to reformulate the system given above with respect to three different observer design approaches, namely, full-order observer, reduced-order observer and the canonical DAE observer. In order to study their sensitivity to high frequency noise, we take the Laplace transform of the estimates in each case, and compare their solutions, do a noise sensitivity analysis and state the conclusions.

2.5.1 Standard approach

full-order observer

The first step is to obtain a standard full-order observer using the estimator equation (1.6), (1.7) and then expressing the results in the Laplace transform domain.

The estimator equation yields

$$\dot{\hat{x}}_1 = \hat{x}_2 + \gamma_1(y - \hat{x}_1) \quad (2.17)$$

$$\dot{\hat{x}}_2 = \gamma_2(y - \hat{x}_1) \quad (2.18)$$

and

$$\hat{y} = \hat{x}_1$$

where $H = [\gamma_1, \gamma_2]^t$, γ_i are positive to guarantee stability. This can be seen by computing the eigenvalues of matrix

$$\begin{bmatrix} -\gamma_1 & 1 \\ -\gamma_2 & 0 \end{bmatrix}$$

The system is stable if the eigenvalues all have negative real part and this occurs if $\gamma_1 > 0$, $\gamma_2 > 0$.

Assuming the initial conditions $\hat{x}_i(0) = 0$ and $\dot{\hat{x}}_i(0) = 0$ for $i = 1, 2$, the Laplace transformation of (2.17) and (2.18) yields

$$s\hat{x}_1^* = -\gamma_1\hat{x}_1^* + \hat{x}_2^* + \gamma_1 y^* \quad (2.19)$$

$$s\hat{x}_2^* = -\gamma_2\hat{x}_1^* + \gamma_2 y^* \quad (2.20)$$

Note, the superscript ' $*$ ' indicates Laplace transform of the variable. That is, $x^* = x^*(s) = \mathcal{L}[x(t)]$

Multiplying (2.19) by s , adding the result to (2.20) and solving for \hat{x}_1^* , the following is obtained:

$$\hat{x}_1^* = \left[\frac{s\gamma_1 + \gamma_2}{s^2 + \gamma_1 s + \gamma_2} \right] y^* \quad (2.21)$$

Substituting for \hat{x}_1^* in (2.20) yields

$$\hat{x}_2^* = \left[\frac{\gamma_2 s}{s^2 + \gamma_1 s + \gamma_2} \right] y^* \quad (2.22)$$

Then, finally, for a standard full-order observer, we obtain the following:

$$\hat{x}^* = \frac{1}{s^2 + \gamma_1 s + \gamma_2} \begin{bmatrix} s\gamma_1 + \gamma_2 \\ \gamma_2 s \end{bmatrix} y^* \quad (2.23)$$

reduced-order observer

Now we obtain a reduced-order observer for this example using the technique developed as in [13], [15] (see Section 1.5). We directly apply the simplified form as shown below:

$$\begin{aligned} \dot{z}_c(t) &= (\bar{F}_{22} - H\bar{F}_{12})z_c(t) + (G_2 - HG_1)u(t) \\ &\quad + (\bar{F}_{21} + \bar{F}_{22}H - H\bar{F}_{12}H - H\bar{F}_{11})z_a(t) \end{aligned} \quad (2.24)$$

$$\hat{z}_b(t) = z_c(t) + Hz_a(t) \quad (2.25)$$

$$\hat{x}(t) = P \begin{bmatrix} y(t) \\ \hat{z}_b(t) \end{bmatrix} \quad (2.26)$$

To compute a reduced-order observer for this example, we set $\bar{F}_{11} = \bar{F}_{21} = \bar{F}_{22} = 0$, $\bar{F}_{12} = 1$, $G_1 = G_2 = 0$, $P = P^{-1} = I$, and $H = \gamma$ (taken as any positive scalar to ensure stability).

$$\dot{z}_c = -\gamma z_c - \gamma^2 y \quad (2.27)$$

$$\hat{z}_b = z_c + \gamma y \quad (2.28)$$

Applying Laplace transformation to (2.27) and (2.28), the following is obtained:

$$sz_c^* = -\gamma z_c^* - \gamma^2 y^* \quad (2.29)$$

where $z_c(0) = 0$, then

$$z_c^* = -\frac{\gamma^2}{s + \gamma} y^* \quad (2.30)$$

$$\hat{z}_b^* = \frac{s\gamma}{s + \gamma} y^* \quad (2.31)$$

Then, our reduced-order observer is

$$\hat{x}^* = \begin{bmatrix} 1 \\ \frac{s\gamma}{s + \gamma} \end{bmatrix} y^* \quad (2.32)$$

Thus $z_b(t)$ is an estimate of $\hat{x}_2(t)$ while $y(t)$ provides $\hat{x}_1(t)$ exactly.

2.5.2 Noise sensitivity analysis

To demonstrate sensitivity to noise in the full-order and reduced-order observers, (2.23) and (2.32), respectively, assume the measurement available for the observer is not exactly $y(t)$, but offset by $\rho(t)$ i.e. $(y(t) + \rho(t))$. Let ρ be a small but high-frequency noise. Then, as to the effect of noise on (2.32), the error in the estimate of $x_1(t)$, $\hat{x}_1(t)$, is small, but the error in the estimate of $x_2(t)$, $\hat{x}_2(t)$ can be very large at high-frequency. This is seen by the following:

since $\tilde{x} = \hat{x} - x$, from (2.32) we obtain

$$\tilde{x}_1 = \rho$$

$$\tilde{x}_2^* = \left[\frac{s\gamma}{\gamma+s} \right] \rho^*$$

We express the frequency, ω , in the argument of a sinusoid given by $\rho(t) = \cos \omega t$ where t designates time. Its Laplace transform is $\rho^*(s) = \frac{s}{s^2 + \omega^2}$.

$$\tilde{x}_2^*(s) = \frac{\gamma s^2}{(s + \gamma)(s^2 + \omega^2)}$$

The right-hand side of the above can be rewritten as a sum of partial fractions as follows

$$= \frac{\gamma^3}{(\omega^2 + \gamma^2)(s + \gamma)} + \frac{\gamma\omega^2 s}{(\omega^2 + \gamma^2)(s^2 + \omega^2)} - \frac{\gamma^2\omega^2}{(\omega^2 + \gamma^2)(s^2 + \omega^2)}.$$

Taking the inverse Laplace transform (using the table in Appendix B) of the above yields

$$\tilde{x}_2(t) = \frac{\gamma^3}{(\omega^2 + \gamma^2)} e^{-\gamma t} + \frac{\gamma\omega^2}{(\omega^2 + \gamma^2)} \cos \omega t - \frac{\gamma^2\omega}{(\omega^2 + \gamma^2)} \sin \omega t$$

For high-frequency, $\omega \rightarrow \infty$, the first and third terms on the right side approach zero. The second term yields $\tilde{x}_2(t) = \gamma \cos \omega t \approx \gamma \rho$ for $t \rightarrow \infty$. Note,

γ must be large for fast convergence of the estimation error. $\tilde{x}_1 = 0$ since \hat{x}_1 is exact, but \hat{x}_2 is affected by high-frequency noise.

Figures 2.1, 2.2 present graphs which are based on the computation of the expression above for $\tilde{x}_2(t)$. They demonstrate that for given high-frequency noise input ω , the amplitudes of error estimates \tilde{x}_2 decrease when γ is decreased. That is, the error estimates of x_2 is reduced for $\gamma \geq \omega$ (see Figures 2.1 and 2.2). However, the error estimates may increase with higher frequencies ω .

In Figure 2.1, γ is set to 5, and the curves are plotted for various ω in the range of $[0.5, 5]$ (for stability, γ are real poles which must be in the left-hand side of the s -plane). In Figure 2.2, $\gamma = 3$ and the ω 's are values from the interval $[1, 5]$. The sinusoidal curves are the error estimates with respect to time in both Figures. The computation of the error estimates yield $|\tilde{x}_2(t)| < 5$ and 3 in Figures 2.1, 2.2, respectively. Note, in Figure 2.1, $\tilde{x}_2(t)$ is approximately zero when $\omega = 0.5$ for $\gamma = 5$. Similarly, we have in Figure 2.2, $\tilde{x}_2(t)$ close to zero when $\omega = 1$ for $\gamma = 3$. However, for other values of ω as shown in their associated Figures, they demonstrate that the error estimates may be too large for $\hat{x}_2(t)$ to serve as a good estimate for state variable $x_2(t)$. The reason will become clearer when the graphs for the full-order case are discussed in a later paragraph.

In the full-order case, when y is replaced by $y + \rho$, we obtain from (2.23) the estimation error

$$\tilde{x}_2^* = \left[\frac{\gamma_2 s}{s^2 + \gamma_1 s + \gamma_2} \right] \rho^*$$

where, as before, $\rho^* = \frac{s}{s^2 + \omega^2}$. However, \hat{x}_1 is not exact. Since $\gamma_i > 0$ and $s^2 + \gamma_1 s + \gamma_2$ can be expressed as a product of two factors $(s + r_1)(s + r_2)$ where the roots $r_i > 0$ where $\gamma_1 = r_1 + r_2$, $\gamma_2 = r_1 r_2$, and $r_1 \neq r_2$. The

Laplace transform expression above yields

$$\tilde{x}_2^* = \frac{\gamma_2 s^2}{(s^2 + \gamma_1 s + \gamma_2)(s^2 + \omega^2)}$$

or

$$\tilde{x}_2^* = \frac{\gamma_2 s^2}{(s + r_1)(s + r_2)(s^2 + \omega^2)}$$

In order to obtain inverse Laplace transform of this expression, we first write the right-hand side in the partial fraction as follows:

$$\frac{A}{s + r_1} + \frac{B}{s + r_2} + \frac{C + Ds}{s^2 + \omega^2}$$

where the unknown constants A, B, C, and D must be computed. The following final expression for $\tilde{x}_2(t)$ is obtained after much algebraic manipulations and inverse transformations:

$$\begin{aligned} \tilde{x}_2(t) = & \left[\frac{r_1^3 r_2}{(r_2 - r_1)(\omega^2 + r_1^2)} \right] e^{-r_1 t} + \left[\frac{r_2^3 r_1}{(r_2 - r_1)(\omega^2 + r_2^2)} \right] e^{-r_2 t} \\ & + \frac{r_1 r_2}{r_2 - r_1} \left[\frac{r_1^3}{\omega^2 + r_1^2} + \frac{r_2^3}{\omega^2 + r_2^2} + 1 \right] \sin \omega t \\ & + \frac{r_1 r_2}{r_1 - r_2} \left[\frac{r_1^2}{\omega^2 + r_1^2} + \frac{r_2^2}{\omega^2 + r_2^2} \right] \cos \omega t \end{aligned}$$

In Figure 2.3, since the real poles, r_i must be in the left-hand side of the s -plane (or negative), we choose $s_1 = -r_1 = -3$, $s_2 = r_2 = -4$ to guarantee stability. The error estimates for $x_2(t)$ are plotted for $\omega \in [1 : 10]$ over t -time interval $(0, \pi)$. It is seen that each curve is plotted while ω is held constant over $t \in (0, \pi)$. The curves are in ascending order and for each ω the state estimate error, \tilde{x}_2 converges to zero along the t axis. Figures 2.1, 2.2 and 2.3 were produced by MATLAB software.

It is interesting to note that expression above is just a low-pass filter applied

to ρ (see [24]). Low-pass filters have the property of smoothing (make ρ less erratic) input signals, making it less sensitive to high-frequency noise. That is, the low-pass filter will effectively attenuate high-frequency noise and transmit the effective signal. Hence, \hat{x}_2 is less sensitive to high-frequency noise in the case of full-order observer (2.23) than in the case of the reduced-order observer (2.32).

The concept of low-pass filter is best described in the following way:

Low-pass filter is a device which readily transmits an input signal that changes slowly as a function of time but does not readily transmit a rapidly changing input signal. Alternatively, a high-pass digital filter transmits a rapidly fluctuating input signal but tends to reject a slowly varying input signal.

2.5.3 Canonical DAE observer approach

We examine the noise sensitivity for the Canonical DAE observer (see equations (2.6)-(2.7)).

Again, using the same example, differentiating and taking the Laplace transformation of the canonical DAE observer (2.6), (2.7), we obtain the following:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 1 \cdot \dot{\lambda} + \gamma_1 \lambda$$

$$\dot{\hat{x}}_2 = 0 + 0 \cdot \dot{\lambda} + \gamma_2 \lambda$$

$$\hat{x}_1 = y$$

where $\hat{x}_1(0) = \hat{x}_2(0) = 0$, $h_{\hat{x}_1}(\hat{x}_1) = 1$, $h_{\hat{x}_2}(\hat{x}_2) = 0$, $\Gamma = (\gamma_1, \gamma_2)$, and $\lambda = \lambda(t)$.

$$s^2 x_1^* = \lambda^* (s^2 + \gamma_1 s + \gamma_2)$$

$$s\hat{x}_2^* = \gamma_2\lambda^*$$

$$y_1^* = \hat{x}_1^*$$

After some algebraic manipulation, we obtain

$$\hat{x}^* = \begin{bmatrix} \hat{x}_1^* \\ \hat{x}_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\gamma_2 s}{s^2 + \gamma_1 s + \gamma_2} \end{bmatrix} y^* \quad (2.33)$$

where γ_1, γ_2 are taken such that its stability is guaranteed.

We see from (2.33) that \hat{x}_1 is directly observed as in the reduced-order observer (2.32), but \hat{x}_2 , which was noise sensitive in the reduced-order case, is just as in the full-order case (2.23). Thus, indicating that the canonical DAE observer does not have the problem of sensitivity to high-frequency noise that the reduced-order observers have and so index one canonical DAE observer formulation is the favored approach

2.5.4 Conclusion

To elaborate further that the canonical DAE observer is certainly the better choice.

We have shown that in (2.32) (the reduced-order observer case), $x_2(t)$ can be very large due to high-frequency noise. That is, by taking for example, $\rho(t) = \cos \omega t$ whose Laplace transform is $\rho^* = \frac{s}{s^2 + \omega^2}$, we obtained $\tilde{x}_2(t) = \gamma \cos \omega t \approx \gamma \rho$ as $t \rightarrow \infty$ and $\tilde{x}_1 = \rho$ is small as assumed. In the case of the full-order observer (2.23), \hat{x}_2 is the same as in canonical DAE observer (2.33) but \hat{x}_1 are quite different as \hat{x}_1 is exact in (2.33) but not so in (2.23), thus

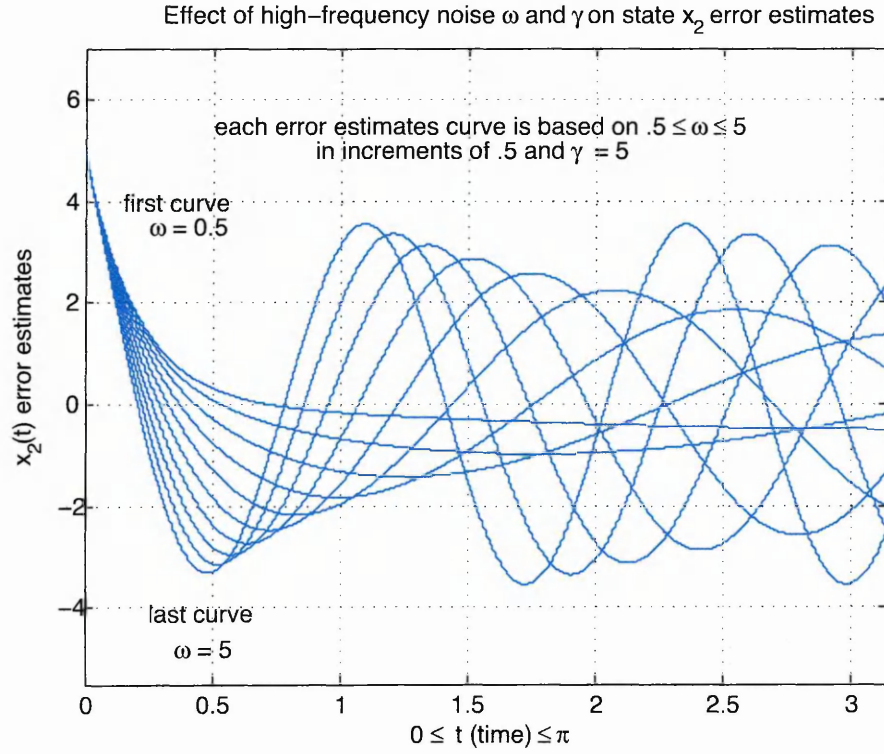


Figure 2.1: Graph of error estimates for full - order observer case

showing an error in its state estimate \hat{x}_1 in (2.23). But, for the canonical DAE observer (2.33), we have shown, using the same $\rho(t)$, that $\tilde{x}_2 = 0$ as $t \rightarrow \infty$, that $\tilde{x}_1 = 0$ ($y = \hat{x}_1$ is exact). Hence, canonical DAE observer is an optimum choice for designing observers.

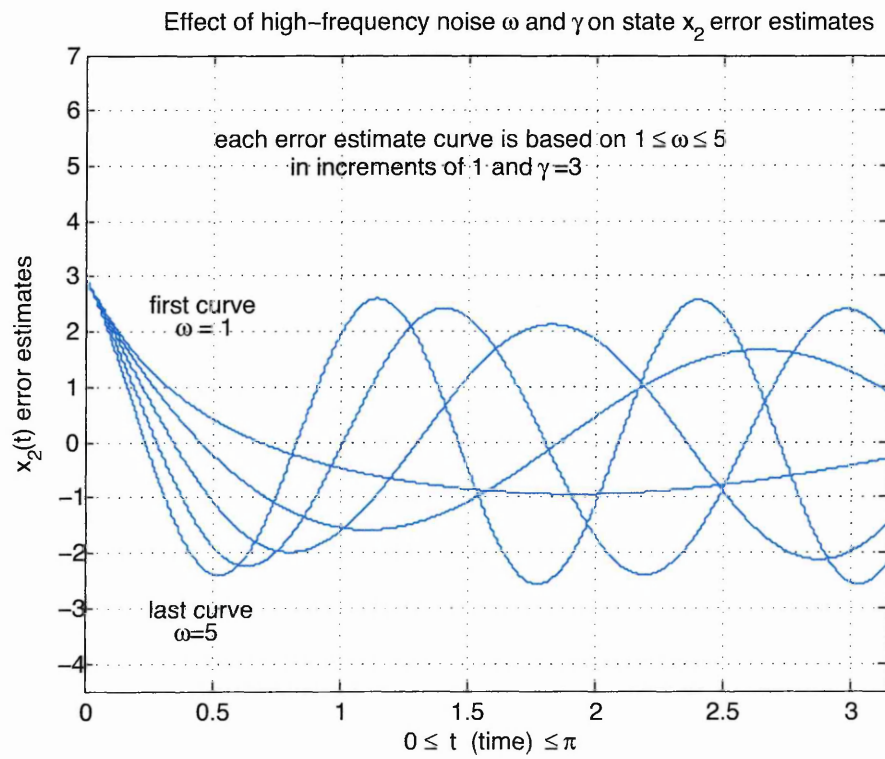


Figure 2.2: Graph of error estimates for reduced - order observer case

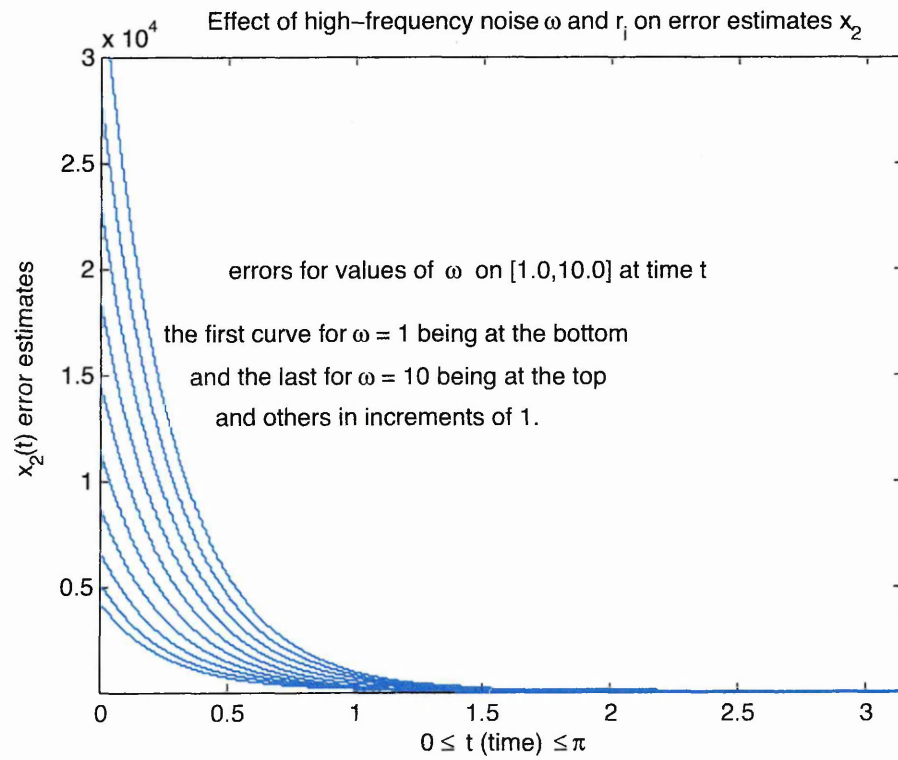


Figure 2.3: Graph of error estimates for canonical DAE observer case

Chapter 3

Estimation Error of Canonical

DAE Observer

3.1 Error Analysis

It is conceivable that the error estimation of the canonical DAE observer equation, depending upon the expressions for $f(\hat{x}, u)$, $\Gamma(\hat{x}, u)$, may not always be expressed in terms of a linear system. Thus, it is necessary to consider

the local behavior of the error around zero, i.e. the equilibrium point at the origin. The ensuing sections discuss the local stability analysis on the error estimation of canonical DAE observer (2.6),(2.7) in the case the error equation does not happen to be a linear system i.e. a nonlinear system. The estimation error is obtained as follows:

The estimation error, $\tilde{x} = \hat{x} - x$, associated with the canonical DAE observer (2.6), (2.7) is

$$\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x} = f(\hat{x}, u) - f(x, u) + h_x(\hat{x})^t \dot{\lambda} + \Gamma(\hat{x}, u) \lambda$$

$$0 = h(\hat{x}) - h(x)$$

where continuous differentiability of f with respect to x and u is assumed. But we have from the Taylor series expansion for a function of one variable the following:

$$f(\hat{x}, u) - f(x, u) = f_x(\hat{x}, u)(\hat{x} - x) + O(\|\tilde{x}\|^2)$$

i.e.

$$f(\hat{x}, u) - f(x, u) = f_x(\hat{x}, u)\tilde{x} + O(\|\tilde{x}\|^2)$$

and,

$$h(\hat{x}) - h(x) = h_x(\hat{x})(\hat{x} - x) + O(\|\tilde{x}\|^2)$$

i.e.

$$h(\hat{x}) - h(x) = h_x(\hat{x})\tilde{x} + O(\|\tilde{x}\|^2)$$

Then the DAE of the error estimate can be expressed in matrix form, where the estimation error \tilde{x} is sufficiently small, yielding

$$\begin{bmatrix} I & -h_x(\hat{x})^t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} f_x(\hat{x}, u) & \Gamma(\hat{x}, u) \\ h_x(\hat{x}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + O(\|\tilde{x}\|^2) \quad (3.1)$$

3.2 Local Stability

This section develops the theory which enables studying the local behavior of system (3.1), in particular, local stability of the nonlinear DAE. The theory demonstrates that local stability can be deduced from stability of the linearization. The form (3.1) is reformulated as a linear time-varying DAE which can be used to establish local stability of the nonlinear equation associated with the canonical DAE observer.

By letting $\zeta = \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}$, we obtain

$$\begin{bmatrix} I & -h_x(\hat{x})^t \\ 0 & 0 \end{bmatrix} \dot{\zeta} = \begin{bmatrix} f_x(\hat{x}, u) & \Gamma(\hat{x}, u) \\ h_x(\hat{x}) & 0 \end{bmatrix} \zeta \quad (3.2)$$

where \tilde{x} is sufficiently small.

On inspecting the elements of the matrices in (3.1) or (3.2), the following assumptions are required to ensure the local stability of the above.

If $x(t)$ denotes the solution of the original system (1.1), (1.2), where the initial condition is x_0 and input function is $u(t)$, the following properties are needed:

1. $f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are bounded in a neighborhood of $(x(t), u(t))$, uniformly in t .
2. $\frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x^2}$ are bounded in a neighborhood of $x(t)$, uniformly in t .
3. $(h_x h_x^t)^{-1}$ and Γ are bounded in a neighborhood of $(x(t), u(t))$, uniformly in t .

Note, higher-order partial derivatives are not needed for linearization.

Before continuing to demonstrate the proof of local exponential stability for the canonical DAE observer, a well-known theorem on exponential stability of the origin of a nonlinear system is cited here.

Theorem[9]: Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(t, x)$ where $f : [0, \infty) \times \bar{D} \rightarrow \mathbb{R}^n$ is continuously differentiable, $\bar{D} = \{x \in \mathbb{R}^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded and Lipschitz on \bar{D} , uniformly in t .

◇

Let

$$\bar{A}(t) = \frac{\partial f}{\partial x}(t, x)|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = \bar{A}(t)x$$

□

The following theorem is used to establish the local stability property of the nonlinear error equation associated with the canonical DAE observer. That is, it will be seen that a certain parameter needs to be determined in order to ensure the stability of this canonical DAE observer.

Theorem I:

Under the assumptions (as enumerated above):

1. $f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are bounded in a neighborhood of $(x(t), u(t))$, uniformly in t
2. $\frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x^2}$ are bounded in a neighborhood of $x(t)$, uniformly in t
3. $(h_x h_x^t)^{-1}$ and Γ are bounded in a neighborhood of $(x(t), u(t))$, uniformly in t ,

if the origin in (3.2) or

$$\begin{bmatrix} I & -h_x(\hat{x})^t \\ 0 & 0 \end{bmatrix} \dot{\zeta} = \begin{bmatrix} f_x(\hat{x}, u) & \Gamma(\hat{x}, u) \\ h_x(\hat{x}) & 0 \end{bmatrix} \zeta$$

where $\zeta = \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}$ and \tilde{x} is sufficiently small,

is exponentially stable, the estimation error \tilde{x} , associated with the canonical DAE observer (2.6),(2.7) or

$$\dot{\hat{x}} = f(\hat{x}, u) + h_x(\hat{x})^t \dot{\lambda} + \Gamma(\hat{x}, u) \lambda$$

$$0 = y - h(\hat{x})$$

converges exponentially to zero provided $\tilde{x}(0)$ and $\lambda(0)$ are sufficiently small.

◇

Proof:

Consider the coordinate transformation: $Q(t) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} = \zeta$, where

$$Q(t) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.3)$$

Let $Q_{11} = I - h_x(\hat{x})^t (h_x(\hat{x}) h_x(\hat{x})^t)^{-1} h_x(\hat{x})$, $Q_{12} = h_x(\hat{x})^t$, $Q_{21} = -(h_x(\hat{x}) h_x(\hat{x})^t)^{-1} h_x(\hat{x})$, $Q_{22} = I$. Then $Q(t)$ is bounded and has a bounded inverse, provided Q_{ij} are bounded for the time trajectory $(\hat{x}, u(t))$. From here on the argument \hat{x} is dropped from $h_x(\hat{x})$.

Then

$$Q = \begin{bmatrix} Q_{11} & h_x^t \\ -(h_x h_x^t)^{-1} h_x & I \end{bmatrix}$$

Note, h_x^t is bounded by assumption 2; also, $(h_x h_x^t)^{-1} h_x$ is bounded by assumptions 2,3. By elementary operations on Q , we see that it is invertible.

In (3.1), let R be the coefficient matrix on the left-hand side of the equal sign

and S the coefficient matrix on the right-hand side of the equal sign. Then, assuming \tilde{x}, λ sufficiently small, (3.1) can be rewritten as

$$RQ\dot{\xi} = (SQ - R\dot{Q})\xi + O(\|Q\xi\|^2) \quad (3.4)$$

where $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ and $Q\xi = \zeta$.

Substituting for R , S and Q in (3.4) and performing algebraic manipulations, yield the following:

$$\begin{aligned} \dot{\xi}_1 &= (f_x Q_{11} + \Gamma Q_{21} - \dot{Q}_{11} + h_x^t \dot{Q}_{21})\xi_1 \\ &+ (f_x Q_{12} + \Gamma Q_{22} - \dot{Q}_{12} + h_x^t \dot{Q}_{22})\xi_2 + O(c \| \xi \|^2) \end{aligned} \quad (3.5)$$

$$0 = h_x Q_{11}\xi_1 + h_x Q_{12}\xi_2 + O(c \| \xi \|^2) \quad (3.6)$$

Since Q is bounded and has a bounded inverse, we can write c as a finite value of norm $\|Q\|^2$ as in the above. That is, $\|Q\xi\|^2 = \|Q\|^2\|\xi\|^2 = c\|\xi\|^2$.

Now (3.5) and (3.6) are represented in matrix form as follows:

$$\begin{bmatrix} \dot{\xi}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{N}(t) & \underline{P}(t) \\ h_x Q_{11} & h_x Q_{12} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + O(\| \xi \|^2) \quad (3.7)$$

where

$$\underline{N}(t) = f_x Q_{11} + \Gamma Q_{21} - \dot{Q}_{11} + h_x^t \dot{Q}_{21} \quad (3.8)$$

and

$$\underline{P}(t) = f_x Q_{12} + \Gamma Q_{22} - \dot{Q}_{12} + h_x^t \dot{Q}_{22} \quad (3.9)$$

where (3.5), (3.6) were simplified with the following:

$$Q_{12} = h_x^t, \dot{Q}_{12} = \dot{h}_x^t$$

$$Q_{22} = I, \dot{Q}_{22} = 0$$

$$Q_{21} = -(h_x h_x^t)^{-1} h_x$$

$$Q_{11} = I + h_x^t Q_{21}$$

$$\dot{Q}_{11} = h_x^t \dot{Q}_{21} + \dot{h}_x^t Q_{21}$$

Thus, all the elements of Q are specified as required. After applying the appropriate substitutions from above to (3.6), we obtain the following:

$$0 = h_x(I + h_x^t Q_{21})\xi_1 + h_x h_x^t \xi_2 + O(\|\xi\|^2)$$

then

$$0 = 0 + (h_x h_x^t)\xi_2 + O(\|\xi\|^2)$$

where $(h_x h_x^t)^{-1}$ is bounded. Hence, $\xi_2 = O(\|\xi_1\|^2)$. That is, as $\|(\xi_1)^2\| \rightarrow 0$, $\frac{|\xi_2|}{\|(\xi_1)^2\|}$ is bounded in the neighborhood of zero.

Now, \dot{h}_x^t can be expressed as

$$\frac{d h_x(x)^t}{dt} \Big|_{x=\hat{x}} = \frac{\partial h_x(x)^t}{\partial x} \Big|_{x=\hat{x}} \dot{\hat{x}}, \quad \dot{\hat{x}} = f(x, u)$$

which is bounded since the first factor is bounded by assumption 2, and the second factor $\dot{\hat{x}}$ is bounded as shown next.

Referring to (2.5) and (1.2), for \tilde{x}, λ sufficiently small, we have as $\lambda, \dot{\lambda} \rightarrow 0$, $v(u, \hat{x}, \lambda, t) \rightarrow 0$. Hence, $\dot{\hat{x}} = f(\hat{x}, u) + O(\|\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}\|)$, where $f(\hat{x}, u)$ is bounded by assumption 1. Hence, $\dot{\hat{x}}$ is bounded. Since all terms in $\underline{N}(t)$ and $\underline{P}(t)$ are accounted for and bounded, then so are $\underline{N}(t)$ and $\underline{P}(t)$.

Then (3.7) is equivalent to

$$\dot{\xi}_1 = \underline{N}(t)\xi_1 + O(\|(\xi_1)\|^2) \tag{3.10}$$

if the linear time-varying system

$$\dot{\xi}_1 = \underline{N}(t)\xi_1 \quad (3.11)$$

is exponentially stable (see Theorem[9] stated previously). The exponential stable system (3.11) is equivalent to the following:

$$\begin{bmatrix} \dot{\xi}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{N}(t) & \underline{P}(t) \\ 0 & h_x h_x^t \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad (3.12)$$

which is really system (3.2) expressed in a coordinate system (see, also equations (3.8),(3.9)). This can be seen, by substituting coordinate system $Q(t)[\xi] = \zeta$ in (3.2). Then, letting R, S, Q be as previously, we obtain

$$RQ\dot{\xi} = R\dot{Q} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + RQ \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = SQ \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

or

$$RQ \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = (SQ - R\dot{Q}) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

After substituting for R, S, Q, \dot{Q} and performing the necessary manipulations on the coefficients above, (3.12) will be obtained; hence, demonstrating the equivalency of (3.12) to (3.2)(see equation (3.4)).

□

The following useful corollary is also established.

Corollary: Suppose the following conditions (1,2,3) (as enumerated above):

1. $f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are bounded in a neighborhood of $(x(t), u(t))$, uniformly in t
2. $\frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x^2}$ are bounded in a neighborhood of $x(t)$, uniformly in t
3. $(h_x h_x^t)^{-1}$ and Γ are bounded in a neighborhood of $(x(t), u(t))$, uniformly in t ,

hold and the origin $\nu = 0$ in

$$\dot{\nu} = [\underline{F}(\hat{x}, u) + \Gamma(\hat{x}, u)\bar{H}(\hat{x})] \nu \quad (3.13)$$

is exponentially stable in the neighborhood of the origin where we let

$$\bar{H}(\hat{x}) = -(h_x h_x^t)^{-1} h_x \quad (3.14)$$

and $\underline{F}(\hat{x}, u)$ defined by

$$\underline{F}(\hat{x}, u) = f_x(\hat{x}, u) \left[I + \left[h_x(\hat{x})^t - \left\{ \frac{\partial h_x(x) f(x, u)}{\partial x} \Big|_{x=\hat{x}}^t \right\} \right] \bar{H}(\hat{x}) \right]. \quad (3.15)$$

Then, the estimation error \tilde{x} , associated with the canonical DAE observer, converges to zero provided $\hat{x}(0) - x(0)$ and $\lambda(0)$ are sufficiently small.

◇

Proof:

If (3.13) is exponentially stable, then

$$\dot{\xi}_1 = (\underline{F}(\hat{x}, u) + \Gamma(\hat{x}, u)\bar{H}(\hat{x}))\xi_1 + O(\|\xi_1\|^2) \quad (3.16)$$

is locally exponentially stable. Now, (3.8) yields

$$\underline{N}(t) = f_x(\hat{x}, u)(I + h_x^t \bar{H}(\hat{x})) - \dot{h}_x^t \bar{H}(\hat{x}) + \Gamma(\hat{x}, u)\bar{H}(\hat{x}) \quad (3.17)$$

after substituting

$$Q_{11} = I + h_x^t Q_{21}, \quad Q_{21} = -(h_x h_x^t)^{-1} h_x = \bar{H}(\hat{x}), \quad \dot{h}_x^t = \frac{\partial h_x(x)^t}{\partial x} \Big|_{x=\hat{x}} f(\hat{x}, u).$$

But this is exactly (3.15) and so we have

$$\underline{N}(t) = \underline{F}(\hat{x}, u) + \Gamma(\hat{x}, u)\bar{H}(\hat{x}) \quad (3.18)$$

which is the coefficient matrix in (3.16). Then, (3.16) is equivalent to (3.10).

□

Also (3.7) \equiv (3.10). Note, (3.13) is exponentially stable around $\nu = 0$ (see also (3.11)).

The discussion in this section is more detailed than in [12]. It is seen from the above that the observer design problem reduces to that of determining Γ that stabilizes (3.13). A useful methodology is covered in the next section.

3.3 Extended Linearization

The list of assumptions in Section 3.2 applies in the following and holds for all $x \in X$ and $u \in U$. In this thesis, Γ selection is based on satisfying the stability criteria related to extended linearization ([1],[2]). These publications actually present theoretical formulation and extension of the gain scheduling technique discussed in [9]. The main idea in this approach is to guarantee local stability of error dynamics of the observer not just around an equilibrium point but for a family of equilibrium points. Despite the fact that this method only guarantees local stability (3.13) around constant operating points (near the equilibrium points being considered), it may provide a systematic procedure and yield a satisfactory observer ([12]), especially, when system inputs are “slowly varying”.

Discussion: Slowly Time Varying Inputs[9]

The system $\dot{x} = f(x, u)$ where $x \in \mathbb{R}^n$, $u = u(t) \in U \subset \mathbb{R}^m$, $\forall t \geq 0$ is considered to be slowly varying if u is continuously differentiable and $\|\dot{u}(t)\|$ is “sufficiently” small. The parameters $u(t)$ could be input variables or time

varying parameters in the analysis of $\dot{x} = f(x, u)$. One usually treats u as a “frozen” u parameter and assumes that for fixed $u = \alpha \in U$ the frozen system has an isolated equilibrium point defined by $x = \underline{h}(\alpha)$. If a property of $x = \underline{h}(\alpha)$ is uniform in α , then it is reasonable to expect that the slowly varying system ($\dot{x} = f(x, u)$) will possess the same property.

The underlying characteristic of such a system is that the motion caused by changes of initial condition is much faster than that caused by inputs or time varying parameters.

Suppose $f(x, u)$ is locally Lipschitz on $\mathbb{R}^n \times U$ and for all $u \in U$ the equation $0 = f(x, u)$ has a continuously differentiable isolated root $x = \underline{h}(\alpha)$; that is, $0 = f(\underline{h}(u), u)$.

In order to analyze stability properties of the frozen equilibrium point $x = \underline{h}(\alpha)$, we can shift it to the origin via the change of variables $z = x - \underline{h}(\alpha)$ to obtain the equation $\dot{z} = f(z + \underline{h}(\alpha), \alpha) \equiv \underline{g}(z, \alpha)$.

The following basic facts about the asymptotic behavior of linear differential equations ought to be kept in mind for the ensuing development.

1. All real or complex solutions of differential equation

$$\dot{\xi} = A\xi(t)$$

satisfy that

$$\xi(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

if and only if all eigenvalues of A have negative real parts.

2. The complex matrix A is said to be Hurwitz if all its eigenvalues have negative real part.

3. A system is said to be observable at time t_0 , if, with the system in state $x(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval.

Definition: $\Omega \subset X \times U$ is called an equilibrium set if all $(x_0, u_0) \in \Omega$ satisfy $f(x_0, u_0) = 0$.

◇

Theorem: Let Ω be an equilibrium set such that for all $(x_0, u_0) \in \Omega$, $(f_x(x_0, u_0), h_x(x_0))$ is observable (see fact 3. above). Then there exists $\Gamma(\hat{x}, u)$ such that the error dynamics associated with the canonical DAE observer is locally stable around every equilibrium point Ω provided \tilde{x}_0 is sufficiently small. Note, The observable condition in this theorem is the same as that required for the existence of a standard observer.

Proof: Let $\underline{J}(\hat{x}, u)$ be an $n \times p$ matrix with entries which are functions of $\hat{x} \in X$ and $u \in U$ such that $f_x(\hat{x}, u) - \underline{J}(\hat{x}, u)h_x(\hat{x})$ is Hurwitz for all $(\hat{x}, u) \in \Omega$, and let

$$\Gamma(\hat{x}, u) = (\underline{J}(\hat{x}, u)h_x(\hat{x}) - f_x(\hat{x}, u))h_x(\hat{x})^t. \quad (3.19)$$

It should be noted that we want a Γ that satisfies the conditions for this theorem for the equilibrium points $(x_0, u_0) \in \Omega$. For such a Γ , we will be able to apply equation (3.16) in Section (3.2). This is shown in the following lemma.

◇

Lemma: Let $(x_0, u_0) \in \Omega$ and consider an equilibrium point $x = x_0$, $\hat{x} = \hat{x}_0$, and $\lambda = \lambda_0$ for (1.1),(1.2),

$$\dot{x} = f(x, u)$$

$$y = h(x)$$

and (2.6),(2.7),

$$\dot{\hat{x}} = f(\hat{x}, u) + h_x(\hat{x})^t \dot{\lambda} + \Gamma(\hat{x}, u) \lambda$$

$$0 = y - h(\hat{x})$$

associated with constant input u_0 , i.e.,

$$0 = f(x_0, u_0) \quad (3.20)$$

$$0 = f(\hat{x}_0, u_0) + \Gamma(\hat{x}_0, u_0) \lambda_0 \quad (3.21)$$

$$y = h(x_0) = h(\hat{x}_0) \quad (3.22)$$

where Γ is defined in (3.19). Then, there exists $\delta > 0$ such that if $\|\hat{x}_0 - x_0\| < \delta$, (3.20)-(3.22) imply that $\hat{x}_0 = x_0$ and $\lambda_0 = 0$.

Proof: Suppose for all $\epsilon < \delta$ there exist \hat{x}_0 , x_0 , and λ_0 satisfying (3.20)-(3.22) such that $\|\hat{x}_0 - x_0\| < \epsilon$, $\tilde{x}_0 = \hat{x}_0 - x_0$, and/or λ_0 is not zero. For ϵ sufficiently small, from (3.20)-(3.22), with $O(\epsilon^2)$ accuracy, and recalling the derivation of equation (3.1); we obtain the following:

$$\begin{bmatrix} f_x(\hat{x}_0, u_0) & \Gamma(\hat{x}_0, u_0) \\ h_x(\hat{x}_0) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ \lambda_0 \end{bmatrix} = 0 \quad (3.23)$$

If the matrix on the left is invertible, the lemma will be proved.

The matrix in (3.23), after using elementary operations and (3.19), can be transformed into the following matrix:

$$\begin{bmatrix} f_x(\hat{x}_0, u_0) - \underline{J}(\hat{x}_0, u_0) h_x(\hat{x}_0) & 0 \\ h_x(\hat{x}_0) & h_x(\hat{x}_0) h_x(\hat{x}_0)^t \end{bmatrix} \quad (3.24)$$

Since the first element in the main diagonal is Hurwitz by construction, it is invertible; and, the second element in the main diagonal is also invertible because h_x has a full row rank. Hence, the lemma is proved.

□

Thus, from the lemma we know that the equilibrium points around which we need to linearize (1.1),(1.2) and (2.6),(2.7) coincide as far as x is concerned, and the equilibrium point λ must be zero in the case of the canonical DAE observer system.

From the corollary in Section 3.2, we have from (3.17) and (3.18) for equilibrium point (\hat{x}_0, u_0) , $\underline{F}(\hat{x}_0, u_0) + \Gamma(\hat{x}_0, u_0)\bar{H}(\hat{x}_0)$, or

$$f_x(\hat{x}_0, u_0)(I - h_x(\hat{x}_0)^t(h_x(\hat{x}_0)h_x(\hat{x}_0)^t)^{-1}h_x(\hat{x}_0)) - \Gamma(\hat{x}_0, u_0)(h_x(\hat{x}_0)h_x(\hat{x}_0)^t)^{-1}h_x(\hat{x}_0)$$

which we need only to check is Hurwitz; however, this is $f_x(\hat{x}, u) - \underline{J}(\hat{x}, u)h_x(\hat{x})$ which is Hurwitz by construction.

□

Hence, matrix $\underline{J}(\hat{x}, u)$ (thus $\Gamma(\hat{x}, u)$) must be selected such that $f_x(\hat{x}, u) - \underline{J}(\hat{x}, u)h_x(\hat{x})$ is Hurwitz for all $(\hat{x}, u) \in \Omega$. Computation of such eigenvalues can be done in a straightforward way in some cases; but, in complex cases, it may be necessary to restrict the domain to Ω to reduce complexity or avoid singularities. A method for constructing $\underline{J}(\hat{x}, u)$ called exact placement of eigenvalues is presented in Ackermann's paper [17], where the method is described for a discrete time-invariant (first-order difference equations). A method for the continuous linear time-invariant dynamic system can be found in [4].

Chapter 4

Interconnected DAE Observers

4.1 Interconnected Systems

The subject of canonical DAE observer design and implementation has been discussed at great length in previous chapters in order that the application of the information derived from this study is utilized to demonstrate the construction of interconnected DAE observer models. This is presented in this section. An example of interconnected observer subsystems is given in affine form (described in Section 1.2) and modeled as a negative feedback system. The observer design is reformulated as an interconnected system of canonical DAE observers (see Section 4.2). The computation of the elements of Γ

required for the canonical DAE observer subsystems using the methodology developed in Chapter 3 is shown. Dynamic systems for both ODE and DAE observers were programmed and executed using the MATLAB & Simulink software package. Graphic results are presented and described in Section 4.8. Section 4.9 provides derivation of analytical solutions to autonomous systems of simultaneous first-order differential equations of observers.

In Section 4.5, the explicit canonical DAE observers of uniform index one are expressed in discrete form and a BDF (backward differentiation formula) DAE solver is briefly discussed. BDF convergence is discussed in Section 4.6. Certain implementation considerations are discussed in Section 4.7.

A simple example is presented here to emphasize the basic concept in using DAE software for the purpose of guaranteeing system stability and solving interconnected observer design problems, which otherwise may be a formidable problem to solve analytically. Section 4.4 introduces the concept of Small - Gain Theorem, but Appendix D should be referenced to see the complexity entailed in designing interconnected systems. The subject matter is covered in this chapter in order to clearly differentiate between the local stability analysis of the error estimation of the state variables and the stability analysis of interconnected observer systems. To the best of the researcher's knowledge, the application of the new DAE observer design approach to interconnected DAE subsystems in the sense of this thesis, has not been published or researched previously.

In analyzing the stability of a nonlinear dynamical system, the order of the system, if possible, is reduced to a lower order system in order to avoid the complexity associated with higher-order systems and to simplify the stability analysis. Hence, in modeling the system, an attempt is made to decompose that system into smaller isolated subsystems by ignoring the interconnec-

tions and analyzing the stability of each system. That is, we combine our conclusion from the above step with information about the interconnections to draw conclusions about the stability of the interconnected system. We illustrate this concept using decomposed ODE observers in affine form which lead to the construction of an interconnected canonical DAE observer of index one.

Examples are given in [9], where the search is done for Lyapunov functions for interconnected systems. However, in this thesis the objective is to devise a methodology for an interconnected canonical DAE observer system which may be implementable using existing DAE solvers. We consider the stability of a proposed interconnection of two observer subsystems (which may or may not be physically realizable or a hypothetical case) with the purpose of demonstrating this methodology. In particular, the case in which the interconnection takes place at the level of inputs and outputs rather than interconnections occurring at the level of states, is considered.

We are primarily concerned with differential systems, i.e. systems where inputs and outputs are related by a set of differential equations. The classification of dynamic element and/or dynamic system relates to elements or system outputs that depend not only on the present value of the output but also on past values. Continuous dynamic elements are assumed to be ordinary differential equations that are an interconnection of continuous dynamic elements. Examples of dynamic systems include motors, ships, networks and other physical entities. A dynamic element of a ship might be the ship's motor, the ship's steering mechanism, etc.

More precisely, we consider two subsystems Σ_1 and Σ_2 , described by a

system of equations of the form

$$\dot{x}^{(i)} = f_i(x^{(i)}, u_i) \quad (4.1)$$

$$y_i = \bar{h}_i(x^{(i)}, u_i) \quad (4.2)$$

where $f_i(0, 0) = 0$ for $i = 1, 2$ (we consider the behavior of the origin as an equilibrium point $(0, 0)$ in a later section) and are locally Lipschitz. For the example to be given below, we assume $x^{(i)} \in \mathbb{R}^2$. Since we also assume $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$, we have

$$\dim(u_2) = \dim(y_1),$$

$$\dim(u_1) = \dim(y_2).$$

The functions $\bar{h}_1(x^{(1)}, u_1)$ and $\bar{h}_2(x^{(2)}, u_2)$ are assumed such that the following constraints are satisfied:

$$u_2 = y_1,$$

$$u_1 = -y_2.$$

Then we have the case of a negative feedback interconnection as shown in Figure 4.2. Note, the negative multiplier (-1) changes the sign of the output in the interconnecting feedback path between subsystems Σ_1 and Σ_2 as shown in Figure 4.2.

We also assume that the nonlinear systems are represented in the affine form

$$f_i(x^{(i)}, u_i) = \bar{f}_i(x^{(i)}) + g_i(x^{(i)})u_i \quad (4.3)$$

$$\bar{h}_i(x^{(i)}, u_i) = h_i(x^{(i)}) + \bar{k}_i(x^{(i)})u_i \quad (4.4)$$

From here on, since $y_i \in \mathbb{R}$, $\bar{k}_i(x^{(i)}) = -k_i$ (negative for conveniency in the following example) where $k_i \neq 0$ is a constant scalar.

The role of constant scalar k_i with respect to an associated interconnected

observer design is described in Sections 4.7 and 4.8.

The type of ODE systems in the following example was created for the purpose of demonstrating that a nonlinear system which may, in practice, be difficult to solve analytically could be reformulated as a canonical DAE observer, another nonlinear system. The linearization technique is then applied to this construction. The solutions of both systems are obtained using appropriate ODE and DAE solvers. The graphs in a later section will demonstrate that the solutions produced very close results, if not exact; thus, the DAE observer design solution checks with that of the original ODE.

In the example given below, the subsystem Σ_i is reconstructed as a canonical DAE observer of index one, where $\hat{x}_2^{(1)}$ is exact and $\hat{x}_1^{(1)}$ is estimated in Σ_1 , and, similarly, $\hat{x}_1^{(2)}$ is exact and $\hat{x}_2^{(2)}$ is estimated in Σ_2 . Each input $u_i \in \mathbb{R}$ associated with its subsystem Σ_i is SISO (single-input single-output). Also, bear in mind, that one of the advantages of canonical DAE observers over reduced-order observers is in not being sensitive to high-frequency noise (as discussed in Chapter 2).

Note, each solver uses a different numerical integrating method (discussed in a later section). Also, the systems are in affine form with unobservable elements $x_1^{(1)}, x_2^{(2)}$.

In order to study observer design problems of this type, we consider, for example, the following second-order subsystems to be used in the interconnection:

Subsystem Σ_1 is given by

$$\dot{x}_1^{(1)} = -x_1^{(1)} + \sin x_2^{(1)} \quad (4.5)$$

$$\dot{x}_2^{(1)} = x_1^{(1)}u_1 + x_2^{(1)} \quad (4.6)$$

$$y_1 = x_2^{(1)} - k_1 u_1 \quad (4.7)$$

Subsystem \sum_2 is given by

$$\dot{x}_1^{(2)} = x_1^{(2)} + x_2^{(2)} u_2 \quad (4.8)$$

$$\dot{x}_2^{(2)} = 8e^{x_1^{(2)}} - x_2^{(2)} \quad (4.9)$$

$$y_2 = x_1^{(2)} - k_2 u_2 \quad (4.10)$$

where the superscript, j , of state variable vectors $x^{(j)} = [x_1^{(j)}, x_2^{(j)}]$ indicate membership of the j^{th} subsystem \sum_j . Note, the above can be rewritten in the form given in (4.1) - (4.2) as follows:

$$\dot{x}^{(1)} = \begin{bmatrix} \sin x_2^{(1)} - x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} + \begin{bmatrix} 0 \\ x_1^{(1)} \end{bmatrix} u_1 \quad (4.11)$$

$$y_1 = x_2^{(1)} - k_1 u_1 \quad (4.12)$$

$$\dot{x}^{(2)} = \begin{bmatrix} x_1^{(2)} \\ 8e^{x_1^{(2)}} - x_2^{(2)} \end{bmatrix} + \begin{bmatrix} x_2^{(2)} \\ 0 \end{bmatrix} u_2 \quad (4.13)$$

$$y_2 = x_1^{(2)} - k_2 u_2 \quad (4.14)$$

In some expressions, the notation is given as in (4.1) and (4.2), where the subscript j of \sum_j will indicate the j^{th} subsystem. Confusion can be avoided by keeping in mind that each subsystem is controlled independently by input supplied by the output of the previous subsystem in the path (see Figure 4.2) and the vectors are

$$x^{(1)} = [x_1^{(1)}, x_2^{(1)}],$$

$$x^{(2)} = [x_1^{(2)}, x_2^{(2)}].$$

For each of the subsystems of the interconnection, for $i = 1, 2$, $x^{(i)} \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{p_i}$, where $n_i = 2$, $p_i = 1$, the functions f_1 , f_2 , h_1 , h_2 are smooth, and

smooth functions y_1, y_2 exist. That is, for each $x^{(1)}, x^{(2)}$, there is a unique pair u_1, u_2 satisfying

$$u_2 = y_1 = \bar{h}_1(x^{(1)}, u_1), \quad (4.15)$$

$$u_1 = -y_2 = -\bar{h}_2(x^{(2)}, u_2). \quad (4.16)$$

Note, from (4.4) for the negative feedback interconnection where $\bar{k}_i(x^{(i)}) = -k_i$, we see that

$$u_2 = h_1(x^{(1)}) - k_1 u_1 \quad (4.17)$$

$$u_1 = -h_2(x^{(2)}) + k_2 u_2 \quad (4.18)$$

This has a unique solution if and only if the matrix,

$$\begin{pmatrix} k_1 & 1 \\ -1 & k_2 \end{pmatrix} \quad (4.19)$$

is invertible. This can be shown by algebraic manipulation in matrix algebra i.e., by solving the determinants of (4.17) and (4.18) for u_1, u_2 and obtaining the denominator $|k_1 k_2 + 1| \neq 0$. Thus, it is nonsingular if $k_1 k_2 \neq -1$.

Note, the nonlinear dynamical system is assumed to be in the affine form as given by (4.11)-(4.12) for system \sum_1 and (4.13)-(4.14) for system \sum_2 where k_i is a nonzero constant scalar. This is a convenience measure in order to facilitate the methodology to be described in this thesis. The expression (4.4), the more complex form suggests a general affine system requiring $\bar{k}_i(x^{(i)})$ with state variable $x^{(i)}$.

As previously mentioned in Section (1.2), the dynamical system studied in this thesis is expressed as $\dot{x} = f(x, u)$, $y = \bar{h}(x, u)$; however, its physical representation may or may not be realizable. Thus, the design of a system with the output given by (4.4) could conceivably be a viable system, but this is beyond the scope of this thesis whose objective is to demonstrate the

technique as developed in this thesis. It will be seen in the graphs provided in a later section that modifying the constant value for k_i may or may not satisfy certain conditions as developed in the case of a negative feedback system described in this thesis. This modification of parameter k_i may serve to demonstrate what could be expected in the case when $\bar{k}_i(x^{(i)})$ is the function as in (4.4).

Note, it is not the intention of this thesis to present a specific physically realizable system in mind where the methodology described serves as a general solution for nonlinear dynamical systems; however, it is conceivable that this innovative concept could be applied to some systems as given by (4.3), (4.4). As shown in Section 4.8, the graphs demonstrate by manipulating the parameter k_i could guarantee the error stability (local) of the interconnected system. Note, the function $\bar{k}_i(x^{(i)})$ could satisfy the stability criteria similar to ((4.30), (4.31)) as developed in Section 4.7 for some interconnected dynamical systems for $\Sigma = \Sigma_1 \circ \Sigma_2$. Perhaps, this could be a topic for further research in this area.

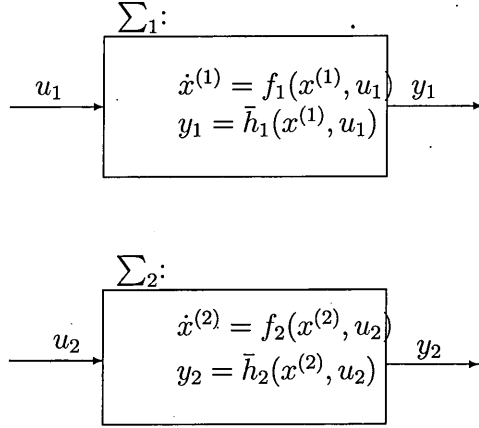


Figure 4.1: Block diagram of each control system Σ_1, Σ_2

4.2 Feedback Interconnection System

Let $x^{(i)} \in \mathbb{R}^2, y_i \in \mathbb{R}, i = 1, 2$, where the subsystems Σ_1, Σ_2 in Figure 4.1 are defined by equations (4.5) - (4.7), (4.8) - (4.10), respectively. Then from the elemental blocks in Fig. 4.1, the negative feedback system is constructed as shown in Figure 4.2. That is, it is possible to express the interconnected system in the following way:

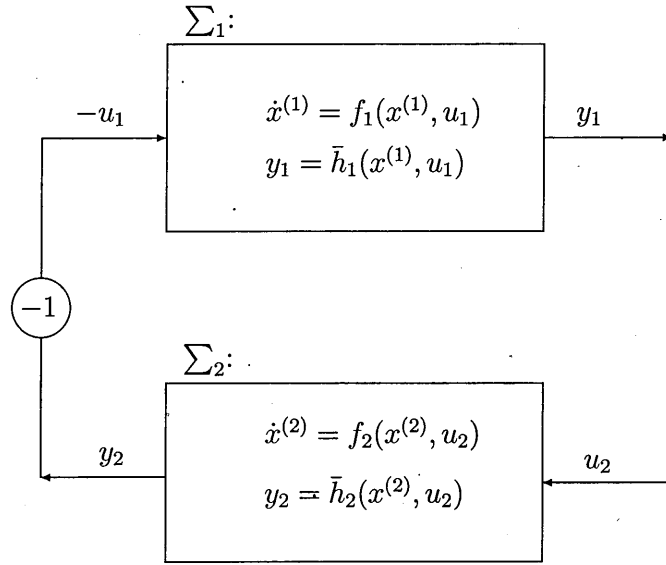


Figure 4.2: Σ - The interconnected system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= \bar{h}(x, u) \end{aligned} \tag{4.20}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^t, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^t$$

or in vector form as:

$$\dot{x}^{(1)} = f_1(x^{(1)}, u_1), \quad y_1 = \bar{h}_1(x^{(1)}, -\bar{h}_2(x^{(2)}, u_2)),$$

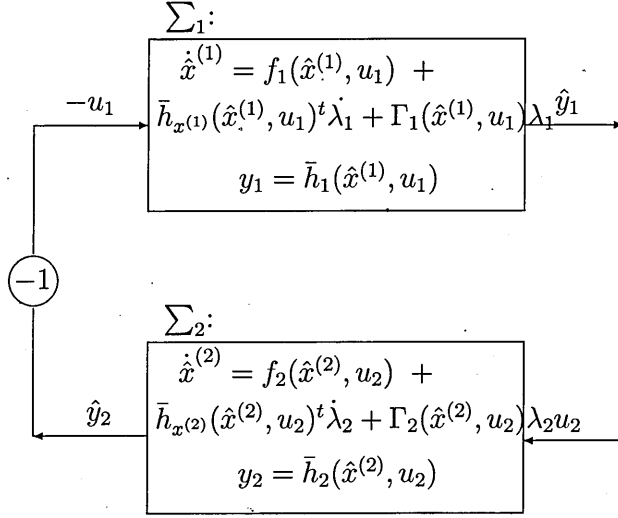


Figure 4.3: Interconnected canonical DAE observer design

$$\dot{x}^{(2)} = f_2(x^{(2)}, u_2), \quad y_2 = \bar{h}_2(x^{(2)}, \bar{h}_1(x^{(1)}, u_1)).$$

Literature and textbooks, e.g. [5], [6], [9] are abundant with observer design methodologies, which are useful in studying the operative stability of systems. To name a few topics, differential geometric approach, Lyapunov-based design, passivity approach, etc. However, in this thesis only the canonical DAE observer type as given by equations (2.6)-(2.7) is considered. Although, it is not the most general observer formulation, it provides a strategy for design and implementation not only of a simple DAE observer but, also an inter-

connected DAE observer system as in Figure 4.3

The canonical DAE observer of subsystem, \sum_i , is represented as follows:

$$\begin{aligned}\dot{\hat{x}}^{(i)} &= f_i(\hat{x}^{(i)}, u_i) + \bar{h}_{x^{(i)}}(\hat{x}^{(i)}, u_i)^t \dot{\lambda}_i + \Gamma_i(\hat{x}^{(i)}, u_i) \lambda_i \\ 0 &= y_i - \bar{h}_i(\hat{x}^{(i)}, u_i)\end{aligned}\quad (4.21)$$

where $\hat{x}^{(i)} \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{p_i}$, $u_i \in \mathbb{R}^{m_i}$, $\lambda_i \in \mathbb{R}$, $\bar{h}_i \in \mathbb{R}^{p_i}$, $f_i \in \mathbb{R}^{n_i}$, $\Gamma_i \in \mathbb{R}^{n_i}$. In particular, for one input, one output, and two state variables $n_i = 2$, $p_i = 1$, $m_i = 1$ for $i = 1, 2$. Note, the above is the interconnected form of the expressions given in (2.6), (2.7) for a single canonical DAE observer.

Then the interconnected DAE observer system is expressed as

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}, u) \\ \hat{y} &= \bar{h}(\hat{x}, u)\end{aligned}\quad (4.22)$$

Using the example for subsystems, \sum_1 and \sum_2 given by (4.11)-(4.14), and, after performing some algebraic manipulations, the expression for each canonical DAE observer subsystem in implicit form is formulated as follows:

For \sum_1 ,

$$0 = \begin{bmatrix} \dot{\hat{x}}_1^{(1)} + \hat{x}_1^{(1)} - \sin \hat{x}_2^{(1)} - \gamma_1^{(1)} \lambda_1 \\ \dot{\hat{x}}_2^{(1)} - \hat{x}_1^{(1)} u_1 - \hat{x}_2^{(1)} - \dot{\lambda}_1 - \gamma_2^{(1)} \lambda_1 \\ y_1 - \hat{x}_2^{(1)} + k_1 u_1 \end{bmatrix} \quad (4.23)$$

where $\hat{x}_2^{(1)}$ is exact.

Similarly, for \sum_2 ,

$$0 = \begin{bmatrix} \dot{\hat{x}}_1^{(2)} - \hat{x}_1^{(2)} - \hat{x}_2^{(2)} u_2 - \dot{\lambda}_2 - \gamma_1^{(2)} \lambda_2 \\ \dot{\hat{x}}_2^{(2)} - 8e^{\hat{x}_1^{(2)}} + \hat{x}_2^{(2)} - \gamma_2^{(2)} \lambda_2 \\ y_2 - \hat{x}_1^{(2)} + k_2 u_2 \end{bmatrix} \quad (4.24)$$

where $\hat{x}_1^{(2)}$ is exact.

From here on, for notation simplification the superscript referencing the subsystems is dropped. The particular \sum_i referenced will be clear from the contents of the discussions. For example, the state vector $x^{(i)} = [x_1^{(i)}, x_2^{(i)}]$ is simplified to $x = [x_1, x_2]$ with reference to \sum_i given.

Note, the qualitative behavior of nonlinear systems (4.5)-(4.6), (4.8)-(4.9) in the vicinity of each equilibrium point could be determined via linearization with respect to that point. However, the following contains a discussion using a similar methodology, extended linearization (Section 3.3), applied to DAE observers subsystems (4.23), (4.24).

It is assumed that the DAE observer of each subsystem \sum_i is index one and that it is easily integrable using standard DAE solver. The error equation for each subsystem has been examined in a previous section, as well as the selection of $\Gamma_i(\hat{x}^{(i)}, u_i)$ that satisfy the stability criterion related to gain scheduling or extended linearization for a family of equilibrium points. That is, the error dynamics equation has been studied where $\tilde{G}_i(\hat{x}^{(i)}, u_i) = f_{x^{(i)}}(\hat{x}^{(i)}, u_i) - \underline{J}(\hat{x}^{(i)}, u_i)h_{x^{(i)}}(\hat{x}^{(i)})$ is Hurwitz by construction and, thus, guaranteeing local stability around every equilibrium point provided $\hat{x}^{(i)}(0) - x^{(i)}(0) = \tilde{x}^{(i)}(0)$ is sufficiently small.

The error equation is expressed for each subsystem by the following:

For \sum_1 , since $\tilde{x}_2^{(1)} = 0$, (4.23) and (4.11) yield

$$\begin{bmatrix} \dot{\tilde{x}}_1^{(1)} \\ \dot{\lambda}_1 \end{bmatrix} = \begin{bmatrix} -1 & \gamma_1^{(1)} \\ -u_1 & -\gamma_2^{(1)} \end{bmatrix} \begin{bmatrix} \tilde{x}_1^{(1)} \\ \lambda_1 \end{bmatrix}.$$

For Σ_2 , since $\tilde{x}_1^{(2)} = 0$, (4.24) and (4.13) yield

$$\begin{bmatrix} \dot{\tilde{x}}_2^{(2)} \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -1 & \gamma_2^{(2)} \\ -u_2 & -\gamma_1^{(2)} \end{bmatrix} \begin{bmatrix} \tilde{x}_2^{(2)} \\ \lambda_2 \end{bmatrix}.$$

Note, in this example the error equation was linear for the canonical DAE observer; however, this may not be true for standard ODE observers.

The Lyapunov Theorem mentioned in Section(2.3) on Convergence Issues is applied here; that is, we find a Lyapunov Function, $\bar{V}(\tilde{x}, \lambda)$ (positive definite) and compute its derivative $\dot{\bar{V}}$ (negative definite) and substitute in the error equations for the derivatives of the arguments, $\dot{\tilde{x}}^{(i)}$, $\dot{\lambda}_i$, for each of the respective subsystem Σ_i .

Let

$$\bar{V}_i = \tilde{x}^{(i)t} \tilde{x}^{(i)} + \lambda_i^2.$$

Then

$$\bar{V}_i = \tilde{x}_1^{(i)2} + \tilde{x}_2^{(i)2} + \lambda_i^2$$

where $\bar{V}_i(0, 0) = 0$, then

$$\dot{\bar{V}}_i = 2[\tilde{x}_1^{(i)} \dot{\tilde{x}}_1^{(i)} + \tilde{x}_2^{(i)} \dot{\tilde{x}}_2^{(i)} + \lambda_i \dot{\lambda}_i].$$

Thus, for Σ_1 , where $\tilde{x}_2^{(1)} = 0$

$$\dot{\bar{V}}_1 = -2[\tilde{x}_1^2 + (u_1 - \gamma_1^{(1)})\lambda_1\tilde{x}_1 + \gamma_2^{(1)}\lambda_1^2]$$

then, since we desire $\dot{\bar{V}}_1$ negative definite, $(u_1 - \gamma_1^{(1)}) = 0$, and $\gamma_2^{(1)} > 0$.

A similar process for Σ_2 where for $\dot{\bar{V}}_2 < 0$, $\tilde{x}_1^{(2)} = 0$, $(u_2 - \gamma_2^{(2)}) = 0$, then $\gamma_1^{(2)} > 0$.

Thus, we should expect local stability of the error dynamics of the observer satisfying the above conditions. It should be borne in mind that this only

guarantees local stability around equilibrium points but it is used in constructing a satisfactory canonical DAE observer.

Next, we demonstrate the construction of $\underline{J}(\hat{x}, u)$, so that $\bar{G}(\hat{x}, u)$ satisfies the Hurwitz property, which in turn generates $\Gamma(\hat{x}, u)$ required to ensure the local stability of the estimation error, \tilde{x} , around only one equilibrium point, (\hat{x}_0, u) . This value is easily obtained from equations (4.11) and (4.13) for each \sum_i . $\bar{G}(\hat{x}, u)h_x(\hat{x}, u)^t$ requires the computation of the Jacobians, $f_x(\hat{x}, u)$, which is also obtained from the original problem stated for each \sum_i . Γ_i is then computed for each subsystem \sum_i :

Define

$$\bar{G}(\hat{x}, u) \equiv f_x(\hat{x}, u) - \underline{J}(\hat{x}, u)h_x(\hat{x}, u).$$

Then substituting (3.19), yields

$$\Gamma(\hat{x}, u) = -\bar{G}(\hat{x}, u)h_x(\hat{x})^t.$$

Now, we compute for subsystem \sum_1

$$\bar{G}_1(\hat{x}, u) = \begin{bmatrix} -1 & \cos \hat{x}_0 \\ u_1 & 1 \end{bmatrix} - \underline{J}h_x(\hat{x}) = \begin{bmatrix} -1 & 1 - \underline{J}_1 \\ u_1 & 1 - \underline{J}_2 \end{bmatrix}$$

where $\underline{J} = \begin{bmatrix} \underline{J}_1 \\ \underline{J}_2 \end{bmatrix}$, $h_x(\hat{x})^t = [0 \ 1]$ and $\cos \hat{x}_0 = 1$ at equilibrium point $(0, 0)$. The Jacobian matrix $f_x(\hat{x}_0, u_1)$ is computed by equating $f(x, u)$ for \sum_1 to zero and finding the equilibrium point, which is $(0, 0)$.

The Jacobian matrix yields

$$f_x(\hat{x}_0, u_1) = \begin{bmatrix} -1 & 1 \\ u_1 & 1 \end{bmatrix}.$$

Then, \underline{J} must be constructed so that \bar{G}_1 is a Hurwitz matrix. The eigenvalues

of each \bar{G}_i are computed from the determinant $|\mu I - \bar{G}_i| = 0$, where μ 's are the eigenvalues. To ensure stability of each subsystem, we must satisfy the property $\text{Re}(\mu_{1,2}) < 0$. The characteristic equation for this subsystem is

$$\mu^2 + \underline{J}_2\mu - ((1 - \underline{J}_2) + (1 - \underline{J}_1)u_1) = 0$$

where the eigenvalues are given by

$$\mu_{1,2} = \frac{-\underline{J}_2 \pm \sqrt{\underline{J}_2^2 + 4((1 - \underline{J}_2) + (1 - \underline{J}_1)u_1)}}{2}$$

Let $\underline{J}_1 = 2$, $\underline{J}_2 = 2$ and $u_1 > 0$, the eigenvalues are complex with negative real parts. Since

$$\Gamma_1(\hat{x}, u) = -\bar{G}_1(\hat{x}, u)h_x(\hat{x})^t = \begin{bmatrix} 1 & \underline{J}_1 - 1 \\ -u_1 & \underline{J}_2 - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we have for subsystem Σ_1 ,

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that $\gamma_1 = 1$ and $\gamma_2 = 1$ (agrees with computation by Lyapunov method that $\gamma_2 > 0$) for Σ_1 and we have $\gamma_i^{(1)} > 0$, $i = 1, 2$.

A similar process is carried out for subsystem Σ_2 .

$$\bar{G}_2(\hat{x}, u) = \begin{bmatrix} 1 & u_2 \\ 8e^{\hat{x}_0} & -1 \end{bmatrix} - \underline{J}h_x(\hat{x}) = \begin{bmatrix} 1 - \underline{J}_1 & u_2 \\ 8 - \underline{J}_2 & -1 \end{bmatrix}$$

where $\underline{J} = \begin{bmatrix} \underline{J}_1 \\ \underline{J}_2 \end{bmatrix}$, $h_x(\hat{x})^t = [1 \ 0]$ and $e^{\hat{x}_0} = 1$ at operating point $(0, 0)$. The equilibrium point is computed by equating $f(x, u)$ for Σ_2 to zero, as was done for Σ_1 .

The Jacobian matrix is given by

$$f_x(\hat{x}_0, u_2) = \begin{bmatrix} 1 & u_2 \\ 8 & -1 \end{bmatrix}$$

as used in the above equation. The characteristic equation for \bar{G}_2 , yields the following roots:

$$\mu_{1,2} = \frac{-\underline{J}_1 \pm \sqrt{\underline{J}_1^2 - 4((\underline{J}_1 - 1) - (\underline{J}_2 - 8)u_2)}}{2}$$

Let $\underline{J}_1 = 2$, $\underline{J}_2 = 9$, and $u_2 > 0$, complex eigenvalues are obtained with $\text{Re}(\mu_{1,2}) < 0$ as desired. Since

$$\Gamma_2(\hat{x}, u) = -\bar{G}_2(\hat{x}, u)h_x(\hat{x})^t = \begin{bmatrix} \underline{J}_1 - 1 & -u_2 \\ \underline{J}_2 - 8 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is obtained for Σ_2 , then

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that $\gamma_1 = 1$ and $\gamma_2 = 1$ (agrees with computation by Lyapunov method that $\gamma_2 > 0$) for Σ_2 and we have $\gamma_i^{(2)} > 0$.

Thus far, the error stability (local) of subsystems Σ_i where $u_1 > 0$ and $u_2 > 0$, respectively, is guaranteed; but, as yet, not that of $\Sigma = \Sigma_1 \circ \Sigma_2$. This will be covered in another section in discussions on Implementation Considerations.

Note, prior to discussing stability of the interconnected DAE observer, we must be clear that the error system developed in a previous section requires that the local behavior of the DAE system (3.1) be stable as discussed in Section 3.2. When we speak of stability of the interconnected DAE observer, we mean the operative stability of the system including the subsystems in Section 4.2, but only over the interval (near the equilibrium point, e.g. $(0, 0)$ in the case above) for which the error equation is stable. That is, it guarantees the operative stability with a high degree of accuracy for that interval of the system's operation. We note the error equation for each subsystem of the interconnected canonical DAE observer (Σ_1, Σ_2) is linear (due to

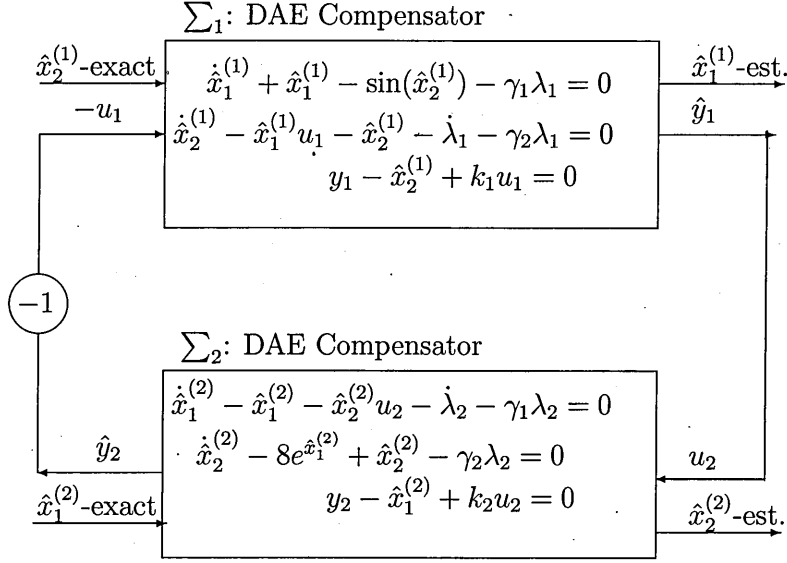


Figure 4.4: Interconnected canonical DAE observer compensators

linearization). For some standard ODE, it would not be linear.

Also, since in the DAE approach the measured states can be substituted by the corresponding outputs, which are common to both the original system and the observer, we may obtain a simpler “more linear” error equation compared to the original system or what could be obtained using standard observers.

We observe that an implicit DAE system such as (4.23), (4.24), have an additional degree of freedom which is useful, in the observer design method

(extended linearization) for each subsystem. However, this is useful regardless of the observer design method used.

Some advantages of the DAE approach that should be noted are as follows:

- Offers a simple strategy for design and implementation of a DAE observer.
- The method guarantees convergence of the observer error for slowly varying input.
- Custom code development is not required because DAE solvers compute the solution of algebraic equations by Newton iteration at every time step.
- Even though the method only guarantees local stabilization of (10), (11) around constant operating points, it provides a systematic design procedure and yields a satisfactory observer.

A variation of the example given could be constructed and modeled as a DAE observer similar to that presented in this Chapter.

For example, consider the negative feedback connection as in Figures 4.2, 4.3 but each subsystem Σ_i is subjected to an additional input $v^{(i)}$, ($i = 1, 2$) representing any external disturbance or noise (unknown/measured) affecting the subsystems Σ_i (see Figures 4.5, 4.6).

This model can be handled in the similar manner as in the example given in Section 4.1 except that the input disturbance $v^{(i)}$ is included.

The ODE and canonical DAE observer are expressed in the general affine form in the following way:

$$\begin{aligned} f_i(x^{(i)}, u_i) &= \bar{f}_i(x^{(i)}) + g_i(x^{(i)})u_i + v^{(i)} \\ \bar{h}_i(x^{(i)}, u_i) &= h_i(x^{(i)}) + \bar{k}_i(x^{(i)})u_i \end{aligned}$$

Since $y_i \in \mathbb{R}$, as previously, we can take $\bar{k}_i(x^{(i)}) = -k_i$ (negative for convenience) where $k_i \neq 0$ is a constant scalar.

The canonical DAE observer of subsystem for the above model, \sum_i , is expressed as follows:

$$\begin{aligned}\dot{\hat{x}}^{(i)} &= f_i(\hat{x}^{(i)}, v^{(i)}, u_i) + \bar{h}_{x^{(i)}}(\hat{x}^{(i)}, u_i)^t \lambda_i + \Gamma_i(\hat{x}^{(i)}, u_i) \lambda_i \\ 0 &= y_i - \bar{h}_i(\hat{x}^{(i)}, u_i)\end{aligned}$$

where $\hat{x}^{(i)} \in \mathbb{R}^{n_i}$, $v^{(i)} \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{p_i}$, $u_i \in \mathbb{R}^{m_i}$, $\lambda_i \in \mathbb{R}$, $\bar{h}_i \in \mathbb{R}^{p_i}$, $f_i \in \mathbb{R}^{n_i}$, $\Gamma_i \in \mathbb{R}^{n_i}$. In particular, $n_i = 2$, $p_i = 1$, $m_i = 1$ for $i = 1, 2$. The functions f_1, f_2, h_1, h_2 are smooth and smooth functions y_i exist. Note, the above is the interconnected form of the expressions given in (2.6), (2.7) for a single canonical DAE observer. Also note, that the above DAE observer is index one.

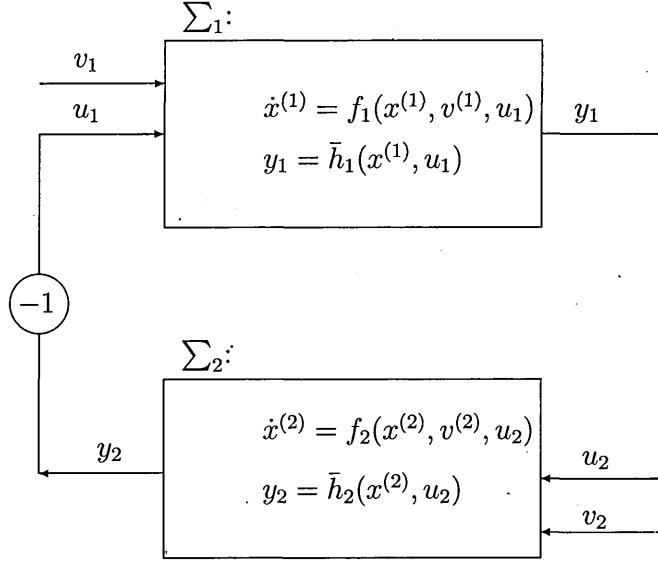


Figure 4.5: Σ - The interconnected system for similar model design

4.3 Interconnection DAE System Stability

The stability of the interconnected DAE subsystems alone does not guarantee stability of the entire interconnected DAE system. That is, feedback connection of two stable systems could be unstable. Even though the subsystems are bounded input-bounded output (BIBO) stable, it is necessary to incorporate some algorithmic mechanism which can be evoked by the DAE solver to ensure the stability of the interconnected system. But first, the concept of the small-gain theorem [9] and its conditions for guarding this

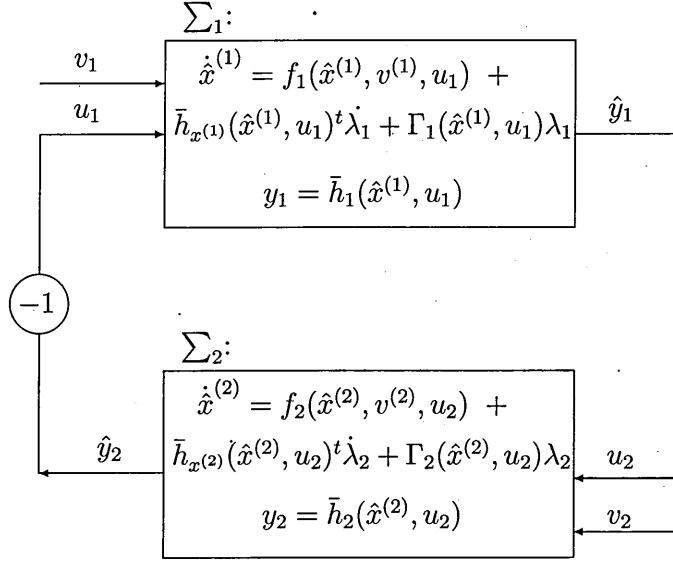


Figure 4.6: Interconnected canonical DAE observer for similar model design

feedback structure against instability will be discussed.

Since the stability of single - input single - output (SISO) nonlinear dynamical subsystem must be ensured, it is desirable to perform a reliable check on the input-output stability (IOS) of the interconnected DAE system by means of an implementable and supplemental software which must be provided by the user. We will discuss the mathematical details and its implementation with the DAE solver in this chapter.

Based on the previous discussion on IOS relative to interconnected DAE sys-

tems, we present a theorem which is formalized as small-gain theorem in [9] and Section 4.4. The concept of gain of a system allows one to track how the norm of a signal increases or decreases as it passes through a system. The small-gain theorem is a general theorem which gives sufficient conditions under which bounded-input produces bounded-output. However, this version of the classical small-gain theorem applies to finite-gain L - stability (see [18]), and is applied to an example of a particular feedback connection (see [9]). This formulation is one in which the question of boundedness is completely disconnected from the questions of existence and uniqueness. The sufficient conditions for existence and uniqueness of solution can be found in [19], and the approach is covered in [18] as well. In [31] the IOS stability is stated in an operational setting without making the role of initial condition explicit. [18] furthers the results in [31] in order to take into account the effects of initial conditions and to express gain function γ of two closed-loop system in terms of the gains of two subsystems. Although this thesis presents an approach to determining input-output stability (IOS) in the case of two interconnected observer system, our major concern is showing that the error system is convergent to the fixed point $(0, 0)$.

4.4 Small-Gain Theorem

A brief view of Small-Gain Theorem might be appropriate here. The concept of Input-Output Stability (IOS) may be useful in studying the stability of interconnections of dynamical systems since the notion of the gain of a system allows us to track how the norm of a signal increases or decreases as it passes through the system (see below for discussion on signal spaces L_p and

extended L_p). This might be helpful in analyzing the feedback connection as in Figure 4.7. However, keep in mind this thesis is primarily concerned with canonical DAE observer designs of interconnected systems.

Some definitions based on functional analysis (any elementary textbook suffices) and control theory are stated as follows (the notations used here are self-contained and explained in this section):

Consider a system characterized by an input/output mapping

$$\dot{y} = Hu$$

where u and y are elements of a normed function space and H is a map between these signal spaces. In particular, u and y will belong to the L_p and extended L_p signal spaces (discussed below).

For each $p = \{1, 2, \dots\}$ the set $L_p[0, \infty) = L_p$ consists of all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ that are measurable (integrable) and

$$\int_0^\infty |f(t)|^p dt < \infty.$$

L_p is a Banach space with regard to the norm

$$\|f\|_p = \sqrt[p]{\int_0^\infty |f(t)|^p dt} < \infty.$$

The set L_∞ consists of all measurable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ that are bounded as

$$\sup_{t \in \mathbb{R}^+} |f(t)| < \infty.$$

L_∞ is also a Banach space with norm

$$\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p = \text{ess} \sup_{t \in [0, \infty)} |f(t)|.$$

Digression(from Real Analysis): A real number M is said to be an essential(ess) bound for a function f whenever $|f(t)| \leq M$ holds for almost all t . A function is called essentially bounded if it has an essential bound. Therefore, a function is essentially bounded if it is bounded except possibly on a set of measure zero. The essential supremum of a function is defined by:

$$\|f\|_{\infty} = \inf(M \geq 0 : |f(x)| \leq M \text{ holds for almost all } x).$$

To introduce the extended signal space L_{pe} , we define the truncation of a signal f .

$$f_T(t) = \begin{cases} f(t) & 0 \leq t < T \\ 0 & t > T \end{cases}$$

Now the set L_{pe} is the set of functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $f_T \in L_p$ for all $T > 0$. We call L_{pe} an extension of L_p because $L_p \subset L_{pe}$. Note that L_{pe} is a linear space but not a Banach space.

These ideas may be extended to multivariable systems by letting $f : \mathbb{R}^+ \rightarrow X$ where X is any finite dimensional linear space. For example, take L_2 whose norm $\|f\|_2$ is associated with an inner product,

$$\langle f, g \rangle = \int_0^{\infty} f(t)g(t)dt.$$

Then, the norm $\|f\|_2 = \langle f, f \rangle^{\frac{1}{2}}$ with respect to this product, becomes a complete inner product space L_2 (also called a Hilbert space).

Next, we discuss causality here. Let U be an m -dimensional linear space with norm $\|\cdot\|_u$ and Y be a p -dimensional linear space with norm $\|\cdot\|_y$. Consider the input signal spaces $L_{pe}(U)$ and $L_{pe}(Y)$ and let $G : L_{pe}(U) \rightarrow L_{pe}(Y)$ be an input-output map between these extended signal spaces.

The map is causal if and only if

$$(G(u))_T = (G(u_T))_T$$

for all $T \geq 0$ and $u \in L_{pe}(U)$. That is, G is causal if and only if for any two inputs $u, v \in L_{pe}$ such that $u_T = v_T$ then $(G(u))_T = (G(v))_T$ for all $T \geq 0$. G is causal if and only if two inputs that are equal over an interval generate outputs that are equal over the same interval. For example, consider the linear operator $G : L_{pe} \rightarrow L_{pe}$ by the convolution integral,

$$(G(u))(t) = \int_0^\infty g(t, \tau)u(\tau)d\tau$$

for some $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Note, that G is causal if and only if $g(t, \tau) = 0$ for $t < \tau$.

A discussion on finite L_p gain follows. The map G is said to have a finite L_p gain if there exists finite constant β_p and γ_p such that for all $T \geq 0$ and $u \in L_{pe}(U)$ that

$$\|(G(u))_T\|_p \leq \gamma_p \|u_T\|_p + \beta_p$$

If $\beta_p = 0$, we say G has finite gain with zero bias. Note, if $G : L_{pe}(U) \rightarrow L_{pe}(Y)$ is causal such that

$$\|G(u)\|_p \leq \gamma_p \|u\|_p + \beta_p$$

then G has a finite L_p gain.

L_p gain of a system is defined as

$$\gamma_p(G) = \inf\{\gamma_p \mid \|(G(u))_T\|_p \leq \gamma_p \|u_T\|_p + \beta_p\}.$$

The input-output map is said to be L_p -stable if and only if it has a finite L_p -gain. Now, we wish to discuss internal stability. Consider the interconnected system in Figure 4.7.

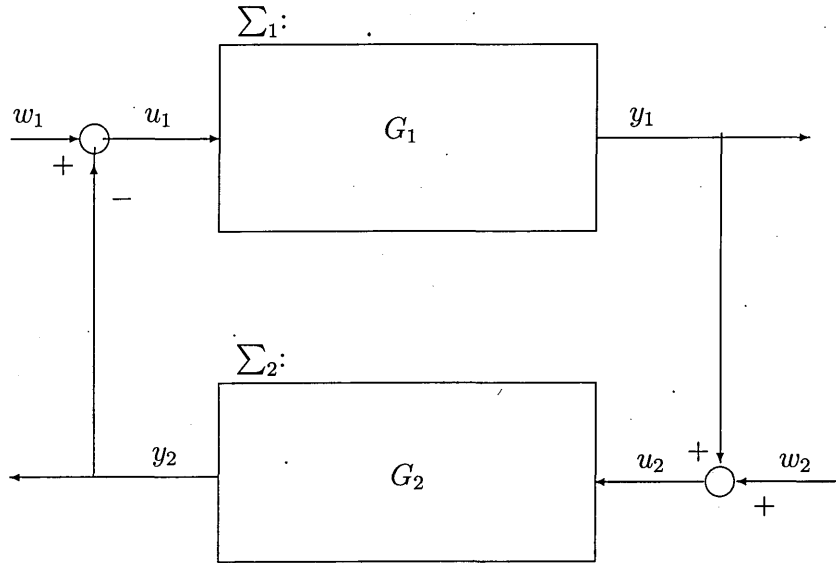


Figure 4.7: Internal Stability

Let $\Sigma(G_1, G_2)$ denote the interconnection of input-output systems

$$G_1 : L_{pe}(U_1) \rightarrow L_{pe}(Y_1)$$

$$G_2 : L_{pe}(U_2) \rightarrow L_{pe}(Y_2)$$

with w_1 and w_2 being inputs to Σ_i that are in L_{pe} , respectively. The system Σ may be described by the following equations:

$$u_1 = w_1 - y_2$$

$$u_2 = w_2 + y_1$$

$$y_1 = G_1(u_1)$$

$$y_2 = G_2(u_2)$$

It is convenient to rewrite this in matrix-vector form as

$$u = w - Fy$$

$$y = G(u)$$

where $u = [u_1, u_2]^t$, $y = [y_1, y_2]^t$, $w = [w_1, w_2]^t$, and

$$F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}.$$

We may write this in terms of two input-output maps in the following way:

$$R_{wu} = \{(w, u) \in L_{pe} | u + FG(u) = w\},$$

$$R_{wy} = \{(w, y) \in L_{pe} | y = G(w - Fy)\}.$$

This closed loop system Σ is said to be internally stable if both R_{wu} and R_{wy} are L_p -stable.

Note, if in Figure 4.7, where $G_i = \Sigma_i$, ($i = 1, 2$), $\Sigma(G_1, G_2)$ is the interconnected system, $u_i \mapsto y_i$ under the appropriate mapping $G_i u_i$, and we set $w_i = 0$, it becomes identical to the interconnected system in Figure 4.2.

This will clarify the need for the checks on bounds of input and outputs to be discussed in a later Section on Implementation Considerations; that is, $\|u_{1\tau}\| = \|y_{2\tau}\|$, $\|u_{2\tau}\| = \|y_{1\tau}\|$ must be appropriately and finitely bounded.

Now, in order to establish the stability of the feedback interconnection in Figure 4.7 (more general than the negative interconnected feedback system in the example), we develop formalism of input-output stability (see [9]). In

Figure 4.7, we have $w_i \neq 0$, and $u_1 = w_1 - y_2$, $u_2 = w_2 + y_1$, where $y_1 = G_1 u_1$ and $y_2 = G_2 u_2$. Hopefully, the following will present a clear discussion on Small Gain Theorem, which is based on Figure 4.7.

This is presented here only to demonstrate a rather difficult methodology to determine the operative stability of the interconnected system. We are not attempting to apply this to the system as given in the original example; however, we apply it to an example in the next few paragraphs.

Keep in mind that as far as this thesis is concerned, the important goal of this thesis is to show that the error system is convergent to the fixed point $(0, 0)$ (as has been demonstrated) which is enough to show that the DAE observer system will converge to give estimates of the unobserved variables in Σ rather than showing that the system Σ is stable.

Consider the system shown above, where $G_1 : L_e \rightarrow L_e$ and $G_2 : L_e \rightarrow L_e$ (p is dropped from subscript of L_{pe} as in the following study). We assume both systems are finite gain stable so that

$$\|y_{1\tau}\| \leq \gamma_1 \|u_{1\tau}\| + \beta_1$$

$$\|y_{2\tau}\| \leq \gamma_2 \|u_{2\tau}\| + \beta_2$$

for all $u_1, u_2 \in L_e$.

We say this feedback loop is well-posed if for every pair of inputs, $w_1, w_2 \in L_e$ there exists a unique output $u_1, y_2, u_2, y_1 \in L_e$. It is desired to know whether the feedback connection from $w \rightarrow u$ and $w \rightarrow y$ is finite gain L – stable.

Small Gain Theorem: The preceding feedback interconnection (see Figure 4.7) is finite gain L – stable if $\gamma_1 \gamma_2 < 1$, where γ_1, γ_2 are finite system gains as in the above equations.

◇

Proof: Define the system

$$Su_{2\tau} = w_{2\tau} + P_\tau(G_1(w_{1\tau} + P_\tau(G_2u_{2\tau})))$$

where

$$P_\tau(w) = \begin{cases} w(t) & t \leq \tau \\ 0 & \tau < t \end{cases}$$

Now consider two signals $u_{2\tau}$ and $\check{u}_{2\tau} \in L_e$ such that

$$\begin{aligned} & \|Su_{2\tau} - S\check{u}_{2\tau}\| \\ &= \|P_\tau G_1(w_{1\tau} + P_\tau(G_2u_{2\tau})) - P_\tau G_1(w_{1\tau} + P_\tau(G_2\check{u}_{2\tau}))\| \\ &= \|P_\tau G_1\| \|P_\tau G_2u_{2\tau} - P_\tau G_2\check{u}_{2\tau}\| \\ &\leq \gamma_1 \|P_\tau G_2u_{2\tau} - P_\tau G_2\check{u}_{2\tau}\| + \beta_1 \\ &\leq \gamma_1 \gamma_2 \|u_{2\tau} - \check{u}_{2\tau}\| + \beta_1 + \beta_2 . \end{aligned}$$

The above is a contraction mapping under the assumption so we can use the contraction mapping principle to infer the existence of a fixed point such that $\bar{u}_{2\tau} = S\bar{u}_{2\tau}$ where

$$\begin{aligned} \bar{u}_{2\tau} &= w_{2\tau} + P_\tau(G_1(w_{1\tau} + P_\tau(G_2\bar{u}_{2\tau}))) \\ &\leq P_\tau(w_2 + G_1(w_1 + G_2\bar{u}_2)) \\ &= P_\tau \bar{u}_2 \in L_e . \end{aligned}$$

A similar argument shows there exists $\bar{u}_{1\tau} \in L_e$ that satisfies the loop equation again.

We have by a previous assumption

$$\|y_{1\tau}\| \leq \gamma_1 \|u_{1\tau}\| + \beta_1,$$

$$\|y_{2\tau}\| \leq \gamma_2 \|u_{2\tau}\| + \beta_2, \quad \forall u_1, u_2 \in L_e, \forall \tau \in [0, \infty).$$

Since we have shown the existence of the solution, we can write

$$u_{1\tau} = w_{1\tau} - (G_2 u_2)_\tau$$

$$u_{2\tau} = w_{2\tau} + (G_1 u_1)_\tau$$

also

$$\|u_{1\tau}\| \leq \|w_{1\tau}\| + \|(G_2 u_2)_\tau\|$$

$$\|u_{2\tau}\| \leq \|w_{2\tau}\| + \|(G_1 u_1)_\tau\|$$

Then, we have

$$\begin{aligned} \|u_{1\tau}\| &\leq \|w_{1\tau}\| + \gamma_2 \|u_{2\tau}\| + \beta_2 \\ &\leq \|w_{1\tau}\| + \gamma_2 (\|w_{2\tau}\| + \gamma_1 \|u_{1\tau}\| + \beta_1) + \beta_2 \\ &= \gamma_1 \gamma_2 \|u_{1\tau}\| + (\|w_{1\tau}\| + \gamma_2 \|w_{2\tau}\| + \beta_2 + \gamma_2 \beta_1) \end{aligned}$$

This implies the bound on $\|u_{1\tau}\|$ is

$$\|u_{1\tau}\| \leq \frac{1}{1 - \gamma_1 \gamma_2} (\|w_{1\tau}\| + \gamma_2 \|w_{2\tau}\| + \beta_2 + \gamma_2 \beta_1).$$

Note, $\gamma_1 \gamma_2 < 1$.

By a similar argument, we can show the following bound

$$\|u_{2\tau}\| \leq \frac{1}{1 - \gamma_1 \gamma_2} (\|w_{2\tau}\| + \gamma_1 \|w_{1\tau}\| + \beta_1 + \gamma_1 \beta_2)$$

which satisfies our definition for L - stability.

If w_1, w_2 are L - stable, the norms $\|w_1\|, \|w_2\|$ are finite. Then $\|u_{1\tau}\|$ is bounded for all τ , uniformly in τ , since $\|w_{i\tau}\| \leq \|w_i\|$ for $i = 1, 2$. Thus, $\|u_1\|$ is finite and L - stable. A similar argument applies to $\|u_2\|, \|y_1\|$, and $\|y_2\|$.

□

Thus, when input $w_i = 0$, it is consistent with the negative feedback system shown in Figure 4.2. Then

$$\|u_{1\tau}\| = \|y_{2\tau}\| = \frac{1}{1 - \gamma_1\gamma_2}(\beta_2 + \gamma_2\beta_1)$$

and

$$\|u_{2\tau}\| = \|y_{1\tau}\| = \frac{1}{1 - \gamma_1\gamma_2}(\beta_1 + \gamma_1\beta_2)$$

are finite.

It should be noted that the in the statement of the Small-Gain Theorem there is no requirement that subsystems G_1, G_2 be L - stable. It simply says that the feedback connection of two input - output stable system, as in Figure 4.7, will be input - output stable provided the product of the system gains is less than one.

We note that the proof of the Small-Gain Theorem is based on the assumption that the subsystems \sum_i are finite gain L -stable. The problem remains as to how one can determine the finite gains γ_1, γ_2 . Discussions relative to this subject matter that might be useful in resolving this question can be found in [9]; however, this study entails several topics which are not considered in this thesis, such as, Hamilton-Jacobi inequality (covered in some text books on Partial Differential Equations, Optimal Control, Fourier transform, Wiener theory, etc.).

The following is a scaled-down discussion to give a little insight into this problem.

Consider the affine systems in u ((4.3), (4.4)) for subsystems \sum_i , where $\bar{k}_i^{(i)} = 0, x_i^{(i)}(0) = 0$. The indices are dropped for conveniency as \sum_i are similar.

Thus, we obtain the following autonomous nonlinear system:

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}$, $u \in \mathbb{R}$; f, g, h are smooth functions and y exists.

Consider the single-input-single-output (SISO) system.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_1^3 - kx_2 + u$$

$$y = x_2$$

then

$$f(x) = \begin{bmatrix} x_2 \\ -ax_1^3 - kx_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(x) = x_2$$

where a and k are positive constants. We use the energy-like Lyapunov function $V(x) = \alpha(\frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2)$ with $\alpha > 0$ as a candidate for the solution of the Hamilton-Jacobi inequality (HJI), which is

$$H(V, f, g, h, \gamma) \equiv \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g(x)^t \left(\frac{\partial V}{\partial x} \right)^t + \frac{1}{2} h(x)^t h(x) \leq 0$$

where $\frac{\partial V}{\partial x} = [\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}]$.

After substituting for the appropriate functions in HJI and by some simple algebraic manipulation, we obtain

$$H(V, f, g, h, \gamma) = (-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2})x_2^2$$

To satisfy HJI, we need to choose $\alpha > 0$ and $\gamma > 0$ such that $-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \leq 0$ or $\gamma^2 \geq \frac{\alpha^2}{2\alpha k - 1}$.

To obtain the smallest possible γ , we choose α to minimize the right-side of the last inequality. The minimum value $\frac{1}{k^2}$ is achieved at $\alpha = \frac{1}{k}$. Thus, choosing $\gamma = \frac{1}{k}$, we conclude that the system is finite-gain L_2 - stable and its L_2 gain is less than or equal to $\frac{1}{k}$. Note, refer to The Small-Gain theorem restated in terms of L_2 - stability in the next few paragraphs.

The solution to the above problem is based on [9]. However, a serious drawback exists in applying this approach, is that it may require a great deal of trial and error effort to find the appropriate Lyapunov function to solve the problem at hand. This could conceivably be the case of the subsystems Σ_1 , Σ_2 , (4.11)-(4.12), (4.13)-(4.14), respectively, in determining finite system gains γ_1 , γ_2 .

Although only the calculation of L_2 - gain of an autonomous system is discussed as given above, some examples in computing finite - gain for L_p and L_∞ systems can be found in references [9], [5]. However, in this thesis our attention is focused on the extended linearization method and the strategy for the design and implementation of a interconnected DAE observer based on this method. This method guarantees convergence of the observer error for slowly varying inputs.

The following example is given using Figure 4.7. Let the system $\Sigma(G_1, G_2)$ described by the following equations:

$$u_1 = w_1 - y_2, \quad u_2 = w_2 + y_1$$

$$y_1 = G_1 u_1, \quad y_2 = G_2 u_2$$

be given by

$$\begin{aligned} y_1(t) &= (G_1 u)(t) = \int_0^\infty e^{-a(t-\tau)} u_1(\tau) d\tau \\ y_2(t) &= (G_2 u)(t) = k y_1(t) \end{aligned}$$

with k, a being given constants. Let $w_2 = 0$ for simplicity. The subsystem represented by G_1 has a transfer function of $\frac{1}{(s+a)}$ while G_2 represents a constant feedback gain of k (s is the Laplace transform variable (see Section 2.5.1)). Then, by control theory, the closed-loop transformation is $\frac{1}{(s+a+k)}$, which corresponds to an impulse response of $e^{-(a+k)t}$. Then, corresponding to an input $u_1(\cdot) \in L_{\infty}$, there exists a unique set of solutions to the system given above $y_1(t), y_2(t)$. $y_1, y_2 \in L_{\infty}$ whenever $u_1 \in L_{\infty}$, and $y_1, y_2 \in L_{\infty}$ whenever $u_1 \in L_{\infty}$, provided $a + k > 0$. Then, the system is L_{∞} -stable. But, if $a + k \leq 0$, then we can find inputs in L_{∞} (e.g. $u_1(t) \equiv 1$) such that the corresponding y_1, y_2 do not belong to L_{∞} . This system is L_{∞} -unstable if $a + k \leq 0$.

Small-Gain Theorem says the feedback connection of two input and two output stable systems as in Figure 4.7 will be input-output stable provided the product of system gains is less than one. This feedback control system is helpful in understanding many results that arise in the study of dynamical systems, especially when feedback is used. Rather than give another specific example as in [9] which requires a background in control theory on the part of the reader, we give a simple development which may help in understanding the principle in terms of space L_p , where $p = 2$ in the following way:

Assuming the usual definition from mathematics for Norms, we state the following informal definitions:

Signal Norms - let signal $\bar{s}(t)$ be a function from $\mathbb{R}^+ \rightarrow \mathbb{R}$. A signal norm measures the size of $\bar{s}(t)$ e.g. 2-norm (energy norm): $\|\bar{s}(t)\| = \sqrt{\int_0^\infty |\bar{s}(t)|^2 dt}$, sup-norm: $\|\bar{s}(t)\|_\infty = \sup_{t \in \mathbb{R}^+} |\bar{s}(t)|$. The space of signals with $\|\bar{s}(t)\|_2 < \infty$

is denoted L_2 .

Gain - given system G with input u , output $y = G(u)$, where G can be a constant, a matrix, or a linear time-invariant system, the gain γ of G should give the largest amplification from u to y .

System Gain - the gain of G is defined as $\gamma(G) = \sup_{u \in L_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in L_2} \frac{\|G(u)\|_2}{\|u\|_2}$
e.g. if $y(t) = \alpha u(t)$ then $\gamma(\alpha) = |\alpha|$.

Bounded-Input Bounded-Output stable (BIBO): G is BIBO stable if $\gamma(G) < \infty$ e.g. if $\dot{x} = Ax$ is asymptotically stable then $G(s) = C(s - A)^{-1}B + D$ is BIBO stable. Note, this is standard in control theory and s is the Laplace transform (see Appendix B for notation) for time t .

Small Gain Theorem - Assume G_1 and G_2 are BIBO stable. If $\gamma(G_1)\gamma(G_2) < 1$ then the closed-loop system from (w_1, w_2) to (u_1, u_2) is BIBO stable. Note, refer to Figure 4.7 in terms of space L_2

◇

Informally, the theorem is proved as follows:

The existence of solution (u_1, u_2) for every (w_1, w_2) has to be verified separately. Then

$$\|u_1\|_2 \leq \|w_1\|_2 + \gamma(G_2)[\|w_2\|_2 + \gamma(G_1)\|u_1\|_2]$$

yields

$$\|u_1\|_2 \leq \frac{\|w_1\|_2 + \gamma(G_2)\|w_2\|_2}{1 - \gamma(G_1)\gamma(G_2)}$$

$\gamma(G_1)\gamma(G_2) < 1$, $\|w_1\|_2 < \infty$, $\|w_2\|_2 < \infty$ yield $\|u_1\|_2 < \infty$.

Similarly, we obtain

$$\|u_2\|_2 \leq \frac{\|w_2\|_2 + \gamma(G_1)\|u_1\|_2}{1 - \gamma(G_1)\gamma(G_2)}$$

so u_2 is also bounded.

□

It should be noted that, if we are assured that given inputs w_1, w_2 there exist unique outputs u_1, u_2, y_1, y_2 (see Figure 4.7) and if we need to know whether the feedback connection is L - stable, the Small-Gain Theorem provides a sufficient condition for finite-gain L - stability of the interconnected system.

The basic concept, which is used in [18], is the input-to-state stability (ISS) property which can be traced through its references. Using certain basic identities and definitions as described Appendix D, this thesis adapts the main results to the problem of interconnected system observer design to demonstrate that a general interconnection of two stable subsystems is an IOS system if an appropriate composition of gain functions is smaller than the identity function. This is established by the “Generalized Small-Gain Theorem”; however, only the portion relevant to the system as given in Figure 4.2 will be described here to demonstrate the complexity entailed in studying such dynamical systems.

The Small-Gain Theorem, in the case of interconnected subsystems as described in previous sections, could be useful; however, this thesis is mainly focused on designing a canonical DAE observer as presented in other chapters. With the availability of powerful DAE solvers the construction of estimate \hat{x} is not very difficult. Although, the only case in which the DAE solvers work as well as ODE solvers is index one, there exist DAE solvers that have been developed for index two and three systems. Perhaps, this could be a subject matter for a future research project.

In the following section, we will discuss the application of the backward differentiation formula (BDF) DAE solver, which is well suited for a uniform index one DAE, to the problem of interconnected canonical DAE observer

(see Figure 4.3). Tracking the inputs-outputs so as to ensure that the interconnected DAE system is IOS will also be described. See Appendix D for more detailed information.

The alternative methodology proposed in handling the problem of interconnected subsystems observer design is presented in the following sections. It will be seen that with the development of canonical DAE observer reconstruction and the availability of DAE solvers, they provide an improved approach in designing observers.

4.5 Numerical Implementation

This section considers the numerical implementation of interconnected canonical DAE observers based on the backward differentiation formula (BDF) method as described in [3], but with time modification.

It is desired to implement the following canonical DAE observer

$$\dot{\hat{x}} = f(\hat{x}, u) + h_x(\hat{x})^t \dot{\lambda} + \Gamma(\hat{x}, u) \lambda \quad (4.25)$$

$$0 = y - h(\hat{x})$$

using existing DAE solvers which are known to be very reliable for certain types of equations such as stiff systems (see [12]). The DAE considered is a system of index one (see Section 2.2), expressed implicitly as

$$F(t, z, \dot{z}) = 0 \quad (4.26)$$

where $z = \begin{bmatrix} \hat{x} \\ \lambda \end{bmatrix}$ and $F_{\dot{z}}$ is singular. Note, $z, F \in \mathbb{R}^{n+p}$.

The first-order BDF method ([3]) is the implicit Euler method which consists of discretizing (4.26) as follows:

$$F(t_{k+1}, z_{k+1}, \frac{z_{k+1} - z_k}{\underline{h}}) = 0 \quad (4.27)$$

where $z_{k+1} - z_k = \begin{bmatrix} \hat{x}_{k+1} - \hat{x}_k \\ \lambda_{k+1} - \lambda_k \end{bmatrix}$ and $\underline{h} = t_{k+1} - t_k$. At each time step, z_{k+1} , is computed in terms of z_k by solving a set of nonlinear algebraic equations (usually by Newton's method which can be found in any elementary textbook on Numerical Analysis). The single step method can be generalized to a multistep method by expressing \dot{z} at $t = t_{k+1}$ by the derivative, evaluated at $t = t_{k+1}$, of a polynomial that interpolates the computed solutions z_j , $j = k+1-n, \dots, k+1$. The multistep BDF DAE solver is particularly well suited for solving uniform index one problems, and, especially, in the case of an interconnected canonical DAE system since it consists of two subsystems \sum_1, \sum_2 (see Figure 4.3) each of uniform index one.

Expressing the canonical DAE observer (4.21) for subsystems, \sum_i as in (4.26) yields

$$F(t, z_i, \dot{z}_i) = \begin{bmatrix} \dot{\hat{x}}_i - f_i(\hat{x}_i, u_i) - (\bar{h}_i)_{x_i}(\hat{x}_i, u_i)^t \dot{\lambda}_i - \Gamma_i(\hat{x}_i, u_i) \lambda_i \\ y_i(t) - \bar{h}_i(\hat{x}_i, u_i) \end{bmatrix}$$

where $z_i = \begin{bmatrix} \hat{x}_i \\ \lambda_i \end{bmatrix}$. Then, under the properties stated in Section 3.2, for \hat{x} sufficiently close to x , the DAE observer (4.21) is uniform index one.

To see this, assume F has a sufficiently smooth solution $z(t)$ satisfying the given initial values, i.e. $z(t_0) = z_0$, and rewrite the subsystem, omitting the i 's for simplicity since \sum_i have the same form.

$$F(t, z, \dot{z}) = \begin{bmatrix} \dot{\hat{x}} - f(\hat{x}, u) - \bar{h}_x(\hat{x}, u)^t \dot{\lambda} - \Gamma(\hat{x}, u) \lambda \\ y(t) - \bar{h}(\hat{x}, u) \end{bmatrix} = 0. \quad (4.28)$$

Then, the above is a DAE and

$$U\dot{z} + Vz = \bar{g}(t)$$

where

$$U = \begin{bmatrix} I & -\bar{h}_x(\hat{x}, u)^t \\ \bar{h}_x(\hat{x}, u) & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & -\Gamma(\hat{x}, u) \\ 0 & 0 \end{bmatrix}, \bar{g}(t) = \begin{bmatrix} f(\hat{x}, u) \\ \dot{y} \end{bmatrix} \quad (4.29)$$

satisfy the *Definitions* and the properties as stated at the end of Section 2.2; thus, since the above equation is an ODE, (4.28) is DAE of uniform index one. Note, the second row of (4.28) is differentiated once with respect to t , yielding $\dot{y} = \bar{h}_x(\hat{x}, u)\dot{\hat{x}}$, and U is nonsingular and of constant rank. Comparing equation (4.29) with equation (2.5), it is seen to be equivalent (with $v(\hat{x}, u, \lambda, t) = \Gamma(\hat{x}, u)\lambda$).

4.6 BDF Convergence

A theorem from [3] is cited, since we are focused only on fully-implicit nonlinear index one systems such as (4.26).

Theorem:

Let (4.26) be a uniform index-one DAE on an interval $I = [t_0, t_0 + T]$. Then the numerical solution of (4.28) by the k -step BDF with fixed stepsize \underline{h} for $k < 7$ converges to $O(\underline{h}^k)$ if all initial values are correct to $O(\underline{h}^k)$ accuracy and if the Newton iteration on each step is solved to $O(\underline{h}^{k+1})$ accuracy.

Note, in order to ensure local stability, we need to assume in the above theorem the properties stated in Section 3.2. Then, we can say that for the solution \hat{x} sufficiently close to x , given initial conditions $x(0)$ and input u ,

the canonical DAE observer (4.28) is uniform index one.

This convergence result has been extended to variable stepsize BDF methods [3]. The convergence and order results for constant and variable stepsize BDF are important because they are used in BDF DAE codes, such as DASSL [3], and has practical application in the case of interconnected systems.

[3] states that it has been shown, that if variable stepsize BDF methods are implemented in such a way that the method is stable for standard ODE's, then k-step BDF method, ($k < 7$) is convergent for fully-implicit index one DAE's.

The correct initial condition for $\hat{x}(0)$ is obtained so that $y(\hat{x}(0), u) = \bar{h}(\hat{x}(0), u)$ and any $\lambda(0)$, in particular, $\lambda(0) = 0$. The choice for stepsize \underline{h} and order k of the integrator depends on several factors related to the performance of the measuring devices (for u and y) i.e. even though they are not user controlled but computed within DASSL, \underline{h} , k depend on the choice of user's specified tolerance so that they make sense with respect to the error of measurement devices.

DAE solvers may require that Jacobian matrices $F_{\hat{z}}$ and F_z be provided. Estimations of these matrices can be computed using (4.28).

Note, for DAE observers \sum_1 (4.23), we have for U, V :

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & -\cos(\hat{x}_2) & -1 \\ -u_1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Similarly, U, V can be computed from (4.24) for \sum_2 . In both cases, $\det(\hat{\rho}U + V)$ is not identically zero as a function of $\hat{\rho}$. Hence, both are regular

pencils and solvable.

4.7 Implementation Considerations

A most widely used production code for DAE's at this time is the code DASSL (Differential Algebraic System Solver) of Petzold [3], which is based on BDF methods. It is designed for solving initial value problems of the implicit form $F(t, z, \dot{z}) = 0$ which are index zero or one. Since the great amount of success in solving scientific and engineering problems has been attained by codes based on BDF, DASSL has benefited from at least thirty years of research and development, and experiences of many users in the area of numerical techniques for ODE and DAE type problems. Since we need to numerically solve uniform index one DAE, $F(t, z, \dot{z}) = 0$, this solver is a very good candidate for solving the interconnected DAE observer problem at hand. Also, since the DAE is index one, the solver does not require any special structure to be provided by the user. Since DASSL source code is public domain software, it is available on internet via Netlib [3]. Also, documentation on DAE Solver can be found by accessing Library Routine Documentation by typing the following URL address, <http://www.win.tue.nl/niconet/NIC2> ([3]). On February 4, 2008, the author has been informed by Dr. R. England that DASSL is available as a built-in function in at least one of two public-domain

versions of MATLAB. However, the DASSL package is not available with MATLAB & Simulink's student version for which I am a licensee. In addition, MATHWORKS, Inc. does not offer software support for this package.

Consider the interconnected DAE system in Figure 4.3 where each subsystem \sum_i , is expressed implicitly in the form $F_i(t, z_i, \dot{z}_i) = 0$ uniform index one and the output function is given in the form (see Section 4.1) as $y_i(\hat{x}, u_i) = h_i(\hat{x}) + k_i u_i$ for $(i = 1, 2)$. The constraints given $u_1 = -y_2$ and $u_2 = y_1$ are required to be satisfied. The implementation interfaces with the DAE solver by calls on DASSL software to solve the DAE equations for each subsystem. Checks on bounds of inputs and outputs within the given required time interval, $I = [t_0, t_0 + T]$ for $\tilde{x} = \hat{x} - x$ sufficiently small and range $t_0 \leq t < T$ are performed. This procedure enables tracking input-output stability of each subsystem at each time step. Solving equations (4.17), (4.18) yields the following relationships: $u_1 = -\Delta^{-1}[h_2(\hat{x}) + k_2 h_1(\hat{x})]$ and $u_2 = \Delta^{-1}[h_1(\hat{x}) - k_1 h_2(\hat{x})]$, where $\Delta^{-1} = \frac{1}{1+k_1 k_2}$; however, for a specific problem, based on u_i 's and γ_i 's, it is necessary to construct \underline{J}_i 's by placement of suitable eigenvalues to enable the determination of the limits on k_i , with respect to $h_i(\hat{x})$, so that the local stability of \sum is guaranteed. Using the given example, we set the inputs to $u_1 > 1$, $u_2 > 1$ (note, since $u_i > 0$ for \sum_i was determined in Section 4.5, u_i can be chosen greater than one) for subsystems \sum_1 and \sum_2 , respectively and, based on equations (4.17), (4.18),

we arrive at the following inequalities:

$$h_1(\hat{x}^{(1)}) > [1 + k_1 u_1] \quad (4.30)$$

$$h_2(\hat{x}^{(2)}) < [-1 + k_2 u_2] \quad (4.31)$$

where $k_i \neq 0$ by assumption (see statement following (4.4)). This can be seen by applying the constraints $y_1 = u_2 > 1$ and $-y_2 = u_1 > 1$ to $y_i(\hat{x}, u) = h_i(\hat{x}) + k_i u_i$ ($i = 1, 2$) i.e. $y_1 = h_1 - k_1 u_1 > 1$ and $-y_2 = -h_2 + k_2 u_2 > 1$ (From here on, $k_i < 0$ is accounted for).

The above conditions must be satisfied so that the stability of the interconnected DAE system $\Sigma = \Sigma_1 \circ \Sigma_2$ within these limits is ensured. Note, that from the construction of $f_x - Jh_x$, the values of the eigenvalues were determined as roots of characteristic polynomials; however, there exist methods for assigning eigenvalues for more complex cases.

Note, the parameters k_i 's appear in the output equations, y_i 's, in each subsystem Σ_i of the interconnected negative feedback system given by equations (4.5) - (4.7) for Σ_1 and equations (4.8) - (4.10) for Σ_2 . We recall that u_i 's are uniquely determined by equations (4.17), (4.18) if and only if (4.19) is invertible. Now, from the engineering observer design perspective, the rationale in prescribing values for k_i 's is to guarantee that conditions (4.30), (4.31) are satisfied for each subsystem Σ_i . Thus, ensuring that the interconnected DAE observer system $\Sigma = \Sigma_1 \circ \Sigma_2$ is a viable and stable system. Since $u_1 = -y_2$, $u_2 = y_1$, it is conceivable that the interconnected observer design

may be dependent upon values of k_i in order to maintain the viability and stability of Σ . Figures 4.8 - 4.19 present graphs for various values of k_i 's for each input u_i .

It is necessary that the implementation interfaces with the DAE solver by calls on DASSL software to solve the DAE equations for each subsystem and performs checks on bounds of inputs and outputs within the finite time interval. That is, after the DAE solver computes the state variables from the differential equations for Σ_i , it is required that $|u_2| = |y_1| < \infty$, $|u_1| = |-y_2| < \infty$ in the range $t_0 \leq t < T$ where the time interval $I = [t_0, t_0 + T]$ is specified and $\tilde{x} = \hat{x} - x$ sufficiently close. This tracking procedure ensures subsystem input-output stability at each time step.

Note, for the given example of the DAE observer design of the interconnected subsystems Σ_1 and Σ_2 , checks on bounds of inputs and outputs $y_i = \bar{h}_i(\hat{x}, u_i)$ (also see inequalities (4.30), (4.31)) must be performed to ensure stability of Σ . It may be necessary to compute and readjust values for k_i , repeating the integration with the changed parameters until the optimum condition for stability is achieved.

The computer algorithm for determining the bounds must be provided by the user. Also, its implementation must be merged with a DAE solver as a callable function.

As stated in Section 4.5, the first order (single step) method used by the

DASSL code replaces the derivative $F_i(t, z, \dot{z})$ with backward difference formula

$$F_i(t_{k+1}, z_{k+1}, \frac{z_{k+1} - z_k}{\underline{h}}) = 0$$

where $\underline{h} = t_{k+1} - t_k$ is the time step. The higher-order methods use higher-order approximations to the derivative.

The resulting system of nonlinear equations for z_{k+1} at each time step is solved by Newton's method [3].

The following parameters are generally required to be provided to DAE solvers:

- 1) Operating interval is given on the time range $I = [t_0, t_0 + T]$, where T is the right-most end of the time interval. On the specified time range, the numerical solution \hat{x}_i should be as stated in the Theorem given in the previous Section with order k and step size \underline{h} as required.
- 2) Initial conditions, $\hat{x}_i(t_0)$, either estimated or known, must be provided. However, DASSL has options for the automatic computation of consistent initial conditions.
- 3) Jacobians $F_z, F_{\dot{z}}$, for \sum_i must be provided; or, they need to be estimated which may possibly slow down the integration and increase the risk of failure. These matrices can be computed using (4.28).
- 4) An initial input value for $u_i(t_0) \neq 0$ for $i = 1, 2$ must be provided, since

u must be persistently exciting to guarantee observability (see discussion in Section 2.3).

5) To compute results with higher accuracy, the relative and absolute tolerances may be specified.

4.8 DAE Solvers in MATLAB and Simulink

This Section discusses the ODE and DAE solutions to the example given in Sections 4.1, 4.2 using ODE and DAE solvers from MATLAB & Simulink software ([21], [22]).

MATLAB & Simulink, a problem solving environment(PSE), contains the capability for solving ODEs and DAEs. It is not available in public domain, as is DASSL; however, it is available to me for a nominal license fee by virtue of the fact that I am currently enrolled at a degree-granting institution. My goal is to determine its relevancy in applying the software to numerically solve the interconnected DAE observer example in this thesis. In this respect, it proved to be successful.

The discussions on the mathematical theory and software developments used for the effective solutions of DAEs of index 1 can be found in [21]. It is conceivable that even though this paper pertains to MATLAB & Simulink,

it may be applicable to other PSEs and, perhaps, applicable to other general scientific computation. A note of interest, MATLAB & Simulink does not solve DAE's of index higher than 1. This software package was chosen by me to solve the constructed models of DAE observer systems given in Sections 4.1, 4.2, instead of applying the DASSL codes written only in FORTRAN language which my computer does not possess the capability to compile and execute. However, MATLAB & Simulink serves to demonstrate the methodology in using a DAE solver to solve a canonical DAE observer system. That is, solve uniform index 1 DAEs in implicit form.

Among the repertoire of functions available to solve ODEs and DAEs of index 1, the following codes of special interest in this thesis are **ode45**, **ode15s**, **ode15i**. **ode45**, a nonstiff solver, was used to solve the original ODEs of subsystems \sum_1 , \sum_2 . A MATLAB function calling on **ode45** codes was developed and graphs of the solutions were produced (see Figures 4.8 through 4.13). Initial conditions given in subsystem \sum_1 are $x_1(0) = 0$, $x_2(0) = 1.1$ and $y_1(0, u_1) = -k_1 u_1$, and $x_1(0) = -1.1$, $x_2(0) = 0$ and $y_2(0, u_2) = -k_2 u_2$ in subsystem \sum_2 .

These graphs present the results obtained from MATLAB & Simulink for the DAE observer constructed in Section 4.1. The output $y_i(x, u_i)$ is based on ODE solution $x = (x_1, x_2)$ using initial conditions as given above. Parameter, k_i , for various input, u_i , are shown in Figures 4.8 through 4.13. When **ode45** was replaced with **ode15s** codes, no appreciable difference in the re-

sults was observed and its results are not presented here. **ode45** is based on an explicit Runge-Kutta of (4, 5) order integration method. It is a one-step solver. That is, in computing $x(t_n)$, it needs only the solution at the immediately preceding time $x(t_{n-1})$. An important characteristic of a stiff system is that the equations are stable, meaning that they converge to a solution. **ode15s** is a variable order solver based on the numerical differentiation formula (NDF). Optionally, it uses the backward differentiation formula (BDF) that are usually less efficient. **ode15i** is used to solve DAEs of index 1. It solves fully implicit differential equations of the form $F(t, x, \dot{x}) = 0$ using the variable order method. The initial conditions must be consistent, meaning $F(t, x_0, \dot{x}_0) = 0$. The graphs of solutions to implicit canonical DAE observers (4.23) for \sum_1 and (4.24) for \sum_2 can be seen in Figures 4.14 through 4.19. Their respective parameters, k_i , and input values, u_i , are shown in these Figures. The system of differential equations is integrated on the interval specified by its initial time and final time. The values of consistent initial conditions are specified. Default values, 1×10^{-3} and 1×10^{-6} , were used for relative and absolute error tolerances, respectively. The conditions computed by (4.30) and (4.31) must be satisfied in order to maintain operating stability of the interconnected systems. That is, for subsystem \sum_1 , where $x_2 > 1$, $y_1 > 1$, it is in time interval $[0, 1]$. For subsystem \sum_2 , where $x_1 < -1$, $y_2 < -1$, it is also over time interval $[0, 1]$. Initial conditions in \sum_1 are set to $x_1(0) = 0$, $x_2(0) = 1.1$. For \sum_2 , $x_1(0) = -1.1$, $x_2(0) = 0$. The output \hat{y}_1

from Σ_1 becomes input u_2 to Σ_2 . Then output \hat{y}_2 from Σ_2 , after negation, becomes the input u_1 to Σ_1 . Both subsystems Σ_i were executed over the same interval of operability where conditions (4.30),(4.31) are met so that $\Sigma = \Sigma_1 \circ \Sigma_2$ is stable. Thus, it is desirable that both subsystems be defined over the same interval in order that we have a consistent system but keeping in mind that since our controller is based on linearization, it guarantees only local error stability.

Note, in Figures 4.8 through 4.13, the graphs of ODE subsystems in (4.5)-(4.10) are based on a priori knowledge of initial condition, $x_i(0)$, input, u_i , and parameter, k_i , used to obtain solutions to implicit DAE solutions as presented in Figures 4.14 through 4.19. Note, we observe that the results for subsystems (4.5) - (4.10) and (4.23) - (4.24) are similar. This is to be expected for this example since the systems are nonstiff; and, thus, the solutions should be very similar if not exact. However, the advantage may be seen in using implicit DAE solvers in the case of stiff systems when needed. The interconnected canonical DAE observer design in our example is presented as interconnected DAE compensators (feedback blocks within the closed-loop of subsystems Σ_i) in Figure 4.4.

In the graphs, Figures 4.8 - 4.19, the output curves y_i are essentially indicators of possible instability of their associated subsystems (Σ_i). It should be noted that the differences seen in the graphs depend upon the input u_i and k_i factors. Their values are indicated on each graph. From the inspection

of the curves in graphs 4.8 - 4.19, it is easy to determine which values of k_i , u_i , the y_i of its respective subsystem satisfy the appropriate conditions (4.30), (4.31). For those y_i 's which do not meet the requirements, we can then suspect that the interconnected DAE observers Σ may not be locally stable.

Basically, we've shown that an observer system of interconnected canonical DAE of uniform index 1 can be solved in their implicit form using **ode15i**. The solutions obtained using the nonstiff solver **ode45** serve as a guide as to the appropriate solutions for the DAE system by comparing its results with that of stiff system solver **ode15i**.

The graphs in Figures F.1, F.2 (Appendix F) present the overall perspective of the ODE and DAE solutions of the combined system $\Sigma_1 \circ \Sigma_2$, where the y_i curves are plotted for values of k_i and u_i as indicated. Keep in mind that although the ODE and DAE solutions are similar, the local error stability is formally guaranteed by using the methodology presented in this thesis for canonical DAE observers but not necessarily in the ODE case. In addition, stiff implicit systems are best handled by DAE solvers such as **ode15i** and **DASSL**.

It should also be kept mind that the model of a interconnected system is simply a decomposition of two complex subsystems Σ_1 , Σ_2 where the outputs are computed based on the appropriate k_i , u_i and x_i . Figures F.1, F.2 (Appendix F) represent the conglomeration of Figures 4.8 - 4.13 of solutions

from a ODE solver, and Figures 4.14 - 4.19 of solutions from a DAE solver, respectively. The observation that should be made is where does y_i satisfy the conditions (4.30), (4.31), which can easily be seen from the graphs. It should not be misconstrued that the graphs represent solutions of a single interconnected system but should be considered as graphs of subsystems for which the y_i 's will guarantee local error stability of the appropriate system Σ when operating with the appropriate parameters, e.g. given u_i , it should answer the question what values of k_i are feasible to use in order that the system $\Sigma = \Sigma_1 \circ \Sigma_2$ operates as a viable one.

Note, by using the results for $\Gamma(\hat{x}, u)$ as computed in Section 4.2, we are assured that the error dynamics associated with the canonical DAE observer is locally stable around each equilibrium point for sufficiently small \tilde{x}_0 . This is based on the concept studied in Chapter 3, i.e. the stability analysis (local behavior) of the estimation error associated with canonical DAE observer. However, in this section we are primarily concerned with the operational stability at and near this equilibrium point of two interacting subsystems which are interconnected in some way. These subsystems are constructed as an interconnected system of canonical DAE observers for which Γ holds. In this Chapter, we present the conditions which ensure local error stability of the entire system Σ .

In Figures 4.20, 4.21, it can be seen that the outputs from the canonical DAE observers converge to the unobserved states of their respective subsystems,

\sum_i . That is, the estimates of state variables, $\hat{x}_i^{(i)}$, converge to the actual state variables, $x_i^{(i)}$, computed by the ODE system. These are based on similar systems used in Figures 4.8 - 4.19. Note, as expected, for the example in Section 4.1, with given parameters k_i and inputs u_i , the results are convergent. The system $\sum = \sum_1 \circ \sum_2$ converges provided conditions (4.30), (4.31) are satisfied at or near the equilibrium point for local error stability.

In furthering the discussion above, the following is a brief review on ensuring that the interconnected DAE system is a viable operating system. The state variable estimates (observed or unobserved) are computed for each \sum_i , and, since it is desired to maintain local error stability, the output y_i (constraint) based on the observed state $x^{(i)}$, parameter k_i and input u_i is also computed. We know from the example (negative feedback interconnection model) in Section 4.1, the inputs $u_1 = -y_2$, $u_2 = y_1$, i.e. u_1 is input to \sum_1 , u_2 is input to \sum_2 , y_1 is output from \sum_1 and y_2 is output from \sum_2 . k_i 's can be set to a value which may ensure that the requirement (4.30), (4.31) is met. Thus, as each \sum_i is subjected to the appropriate input u_i , parameter k_i is a constant factor that may be modified or selected so that the interconnected system's (\sum) local error stability is maintained. That is, if need be, the k_i can be changed as the u_i is passed to the appropriate \sum_i .

Depending on specified characteristics of the feedback, it may be necessary to modify parameter k_i in order to achieve the operating condition. To see this, the graphs show for \sum_i for both ODE and DAE systems the effect of

the change in k_i given u_i . The MATLAB & Simulink codes for Figures 4.8 - 4.19, F1 and F2 do not reflect the check for the condition (4.30), (4.31); however, the graphs for u_i serve to demonstrate that different y_i curves computed based on k_i can be seen by inspection. The same objective is, conceivably, obtainable with own codes to handle the test for condition (4.30), (4.31).

4.9 Analytical Solutions

This section contains the derivation of analytical solutions to subsystem \sum_1 given by equations (4.5)-(4.6) and subsystem \sum_2 given by (4.8)-(4.9). The graphs associated with each system are plotted in this section. In Section 4.8, the ODE system of equations (4.5)-(4.7) and (4.8)-(4.10) and the DAE system of estimators (4.23),(4.24) are calculated using ODE and DAE solvers in MATLAB & Simulink, respectively. The solutions and graphs in Section 4.8 can be compared with those in this section.

For subsystem \sum_1 , we see that (4.5),(4.6) are simultaneous first-order differential equations in which both derivatives \dot{x}_1, \dot{x}_2 (dropping superscripts since we are dealing with only one subsystem at a time) are autonomous and the independent variable time t appears only in the form of derivatives. Thus,

we formulate

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{x_2 + u_1 x_1}{\sin x_2 - x_1}$$

where $u_1 \neq 0$.

We solve the above by integrating the following

$$(\sin x_2 - x_1)dx_2 = (x_2 + u_1 x_1)dx_1 ,$$

or

$$u_1 x_1 dx_1 + d(x_1 x_2) = \sin x_2 dx_2$$

and obtaining

$$\frac{u_1 x_1^2}{2} + x_1 x_2 = -\cos x_2 + C_1$$

Applying initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$, the integrating constant C_1 becomes $\cos(1.1)$ or .4535961.

Then, by completion of squares we obtain,

$$(x_1 + \frac{x_2}{u_1})^2 = (\frac{x_2}{u_1})^2 + \frac{2}{u_1} [.4535961 - \cos x_2]$$

Solving for x_1 yields,

$$x_1 = \frac{1}{u_1} (-x_2 \pm \sqrt{(x_2)^2 + 2u_1 [.4535961 - \cos x_2]}) \quad (4.32)$$

Similarly for \sum_2 , writing equations (4.8), (4.9) in the autonomous form

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{8 e^{x_1} - x_2}{x_1 + x_2 u_2}$$

where $u_2 \neq 0$. Then integrating the above yields

$$u_2 \frac{x_2^2}{2} + x_1 x_2 = 8 e^{x_1} + C_2$$

Using the initial conditions $x_1(0) = -1.1$ and $x_2(0) = 0$ in the above equation, the integrating constant $C_2 = -2.662968$ is obtained. Then, as before, by completion of squares and solving for x_2 , we obtain

$$x_2 = \frac{1}{u_2}(-x_1 \pm \sqrt{(x_1)^2 + 16u_2[e^{x_1} - .332871]}) \quad (4.33)$$

and $u_2 \neq 0$.

The graph of the analytical solution of equation 4.32 (using the positive square root) is given in Figures 4.22, 4.24 and of equation 4.33 (using negative square root) in Figures 4.23, 4.25. It should be noted that the constraint equations are not plotted. As can be seen from the graphs in Figures 4.8 - 4.19, F.1 and F.2, the state variables $x_i^{(i)}$ depend on the initial conditions and the constraints y_i 's are dependent upon the parameters k_i and input u_i associated with each subsystem. In Figures 4.22, 4.23, 4.24, 4.25 for the analytic solutions, the time increment is set to 1×10^{-4} over time interval $[0, 1]$. In Figures 4.22, 4.24 the analytical solution to equation 4.32 (subsystem \sum_1), x_1 depends on the independent variable x_2 and, similarly, for \sum_2 (Figures 4.23, 4.25), the analytical solution to equation 4.33 (subsystem \sum_2) x_2 depends on x_1 . These are applied to the MATLAB functions developed to compute the analytical solutions. The results of the graphs in Figures 4.22, 4.24 are clearly comparable to Figures 4.8, 4.14, respectively. Similarly, Figures 4.23, 4.25 are comparable to Figures 4.9, 4.15, respectively. Note, in the graphs (Figures 4.22 - 4.25) tracking of the unobserved states are as shown. That is, in Figures 4.22, 4.24 (subsystem \sum_1), the unobserved

states, x_1, \hat{x}_1 are tracked, respectively. Similarly, in Figures 4.23, 4.25 (subsystem Σ_2), the unobserved states, x_2, \hat{x}_2 are tracked, respectively.

Briefly recapitulating:

The original interconnected ODE observer system was expressed in the affine form and then reformulated as an index 1 interconnected canonical DAE observer. The ODE system was solved using **ode45** and the DAE system with **ode15i** and the solutions were found to be similar as the dynamical ODE and DAE systems from our example are essentially nonstiff. However, there may be advantages to be gained in using the implicit DAE solver which handles stiff systems and does not affect the output y_i of affine form. The ODE solution serves as a guide to the appropriate solutions from the DAE solver. The intent here is to show that this advantage may be attained for nonlinear systems with, perhaps, nonlinear output as a system of differential-algebraic equations of index 1.

It must be reemphasized that the local stability issue of each connected subsystem operates independently from others, and the conditions of the inequality constraints must be satisfied with respect to its own subsystem Σ_i . However, the problem is in determining the input $u_i(t)$ based on y_i which guarantees the local stability for $\Sigma = \Sigma_i \circ \Sigma_2$. This is depicted in Figures F.1 and F.2. For your reference, Appendix F contains MATLAB codes implemented and used by author in order to produce graphics.

The Following Remark is Stated:

If the convergence of the canonical DAE observer (\sum_i) is not guaranteed, then the problem cannot be solved since there may exist $u_i \in U$ such that \sum does not converge.

The design engineer whether he is using MATLAB & Simulink or DASSL is required to provide his automated checks on input and output bounds in order to maintain a viable system over the performance interval.

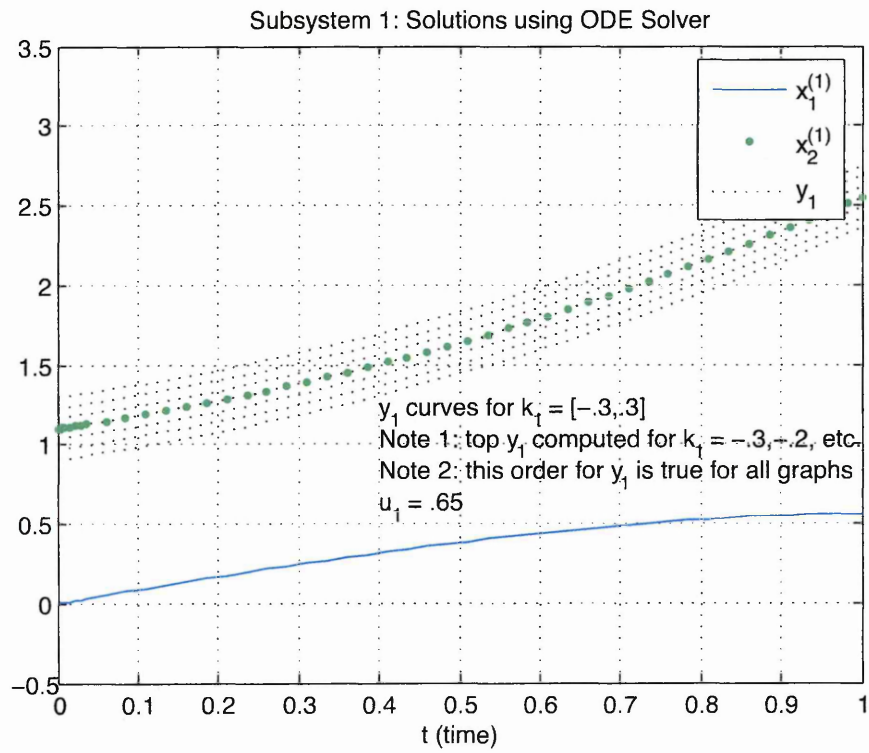


Figure 4.8: Graphs for ODE (4.5)-(4.7) (subsystem Σ_1), for input $u_1 =$

.65, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$,

$$y_1(0, u_1) = -k_1 u_1$$

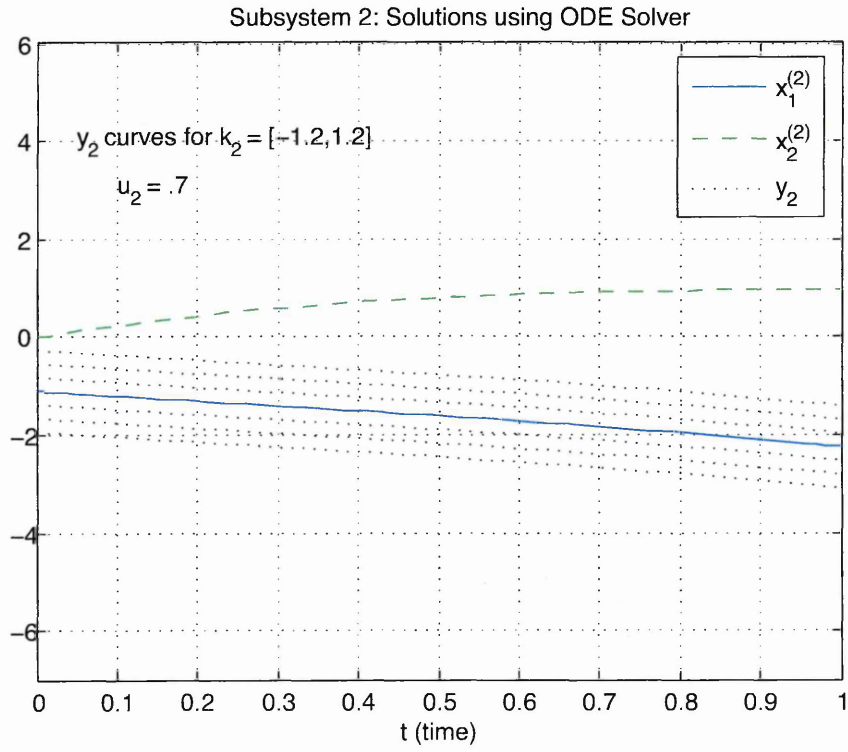


Figure 4.9: Graphs for ODE (4.8)-(4.10) (subsystem Σ_2), for input $u_2 = .7$,

parameters $k_2 = [-1.2, 1.2]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$,

$$y_2(0, u_2) = -k_2 u_2$$

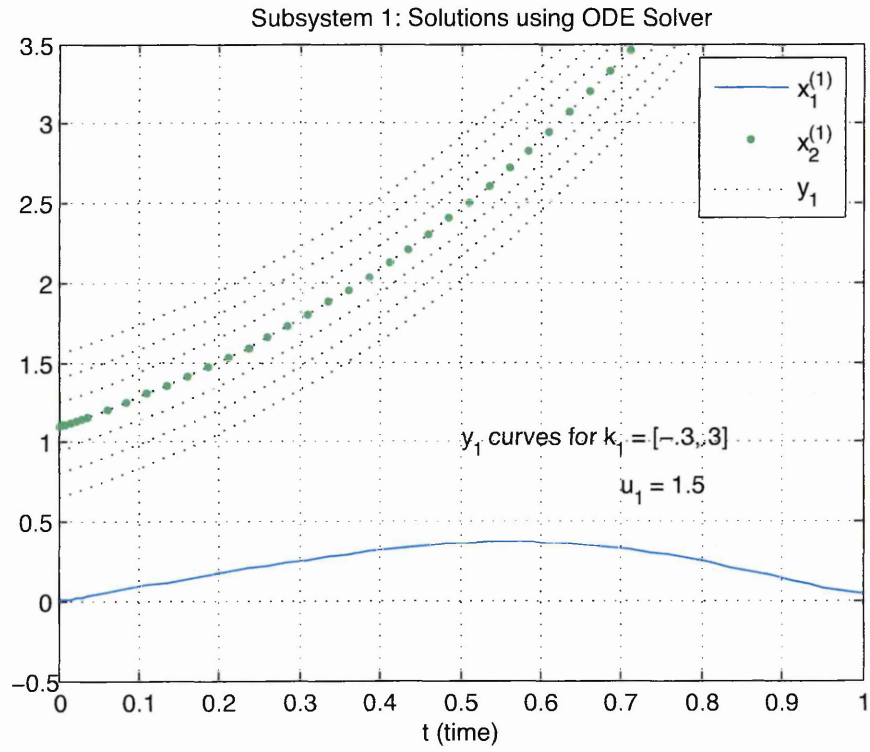


Figure 4.10: Graphs for ODE (4.5)-(4.7) (subsystem Σ_1), for input $u_1 =$

1.5, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$,

$$y_1(0, u_1) = -k_1 u_1$$

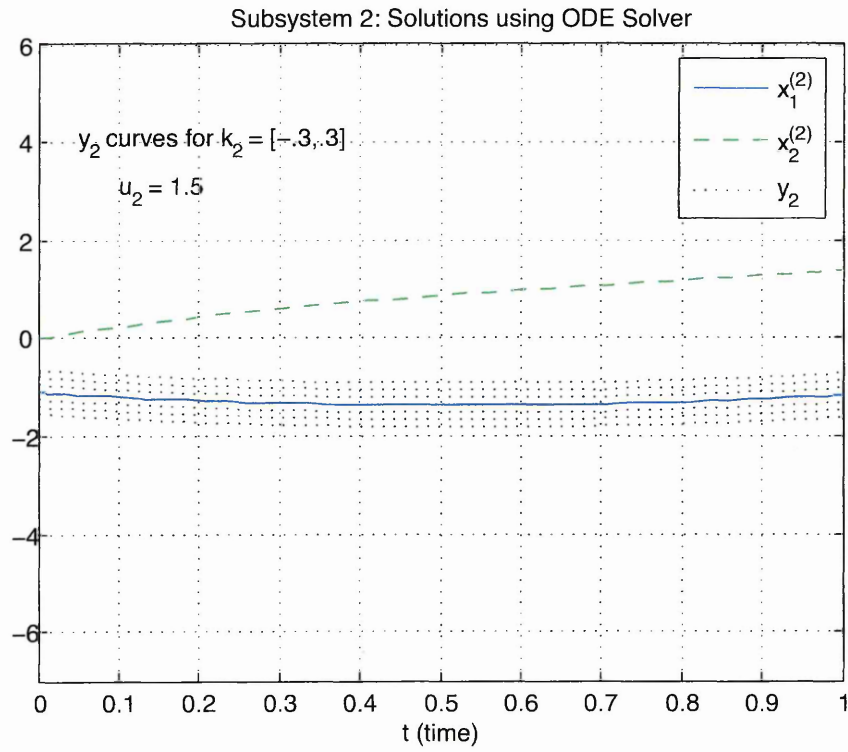


Figure 4.11: Graphs for ODE (4.8)-(4.10) (subsystem Σ_2), for input $u_2 =$

1.5, parameters $k_2 = [-.3, .3]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$,

$$y_2(0, u_2) = -k_2 u_2$$

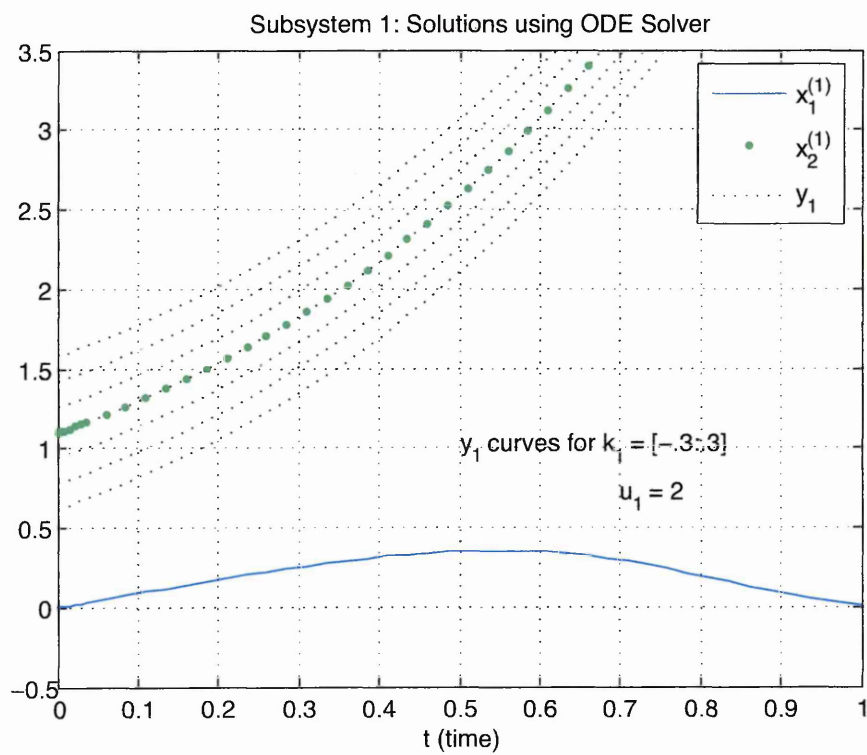


Figure 4.12: Graphs for ODE (4.5)-(4.7) (subsystem Σ_1), for input $u_1 = 2$,

parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$,

$$y_1(0, u_1) = -k_1 u_1$$

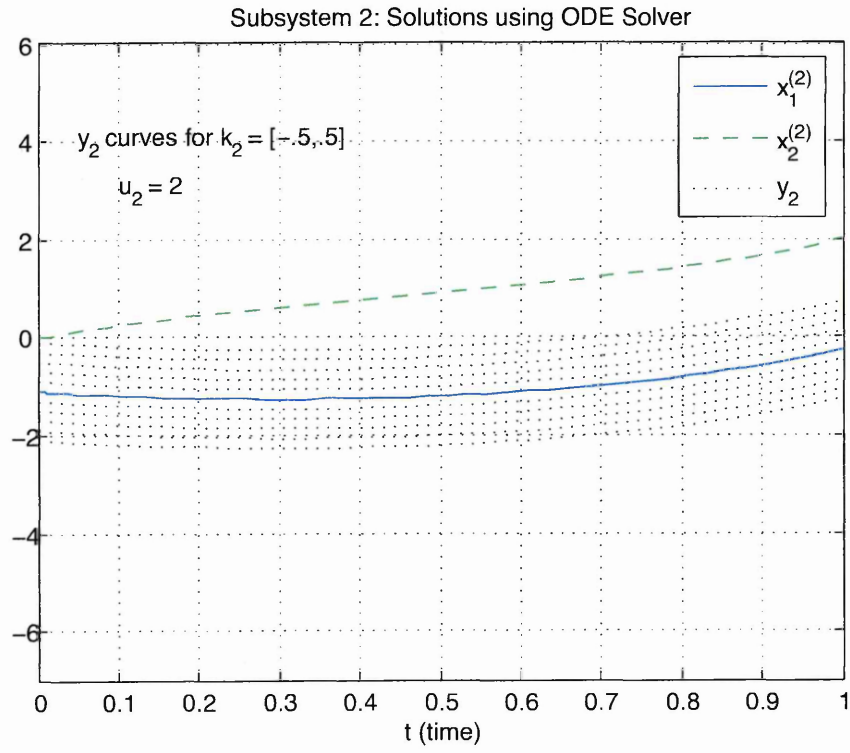


Figure 4.13: Graphs for ODE (4.8)-(4.10) (subsystem Σ_2), for input $u_2 =$

2, parameters $k_2 = [-.5, .5]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$,

$$y_2(0, u_2) = -k_2 u_2$$

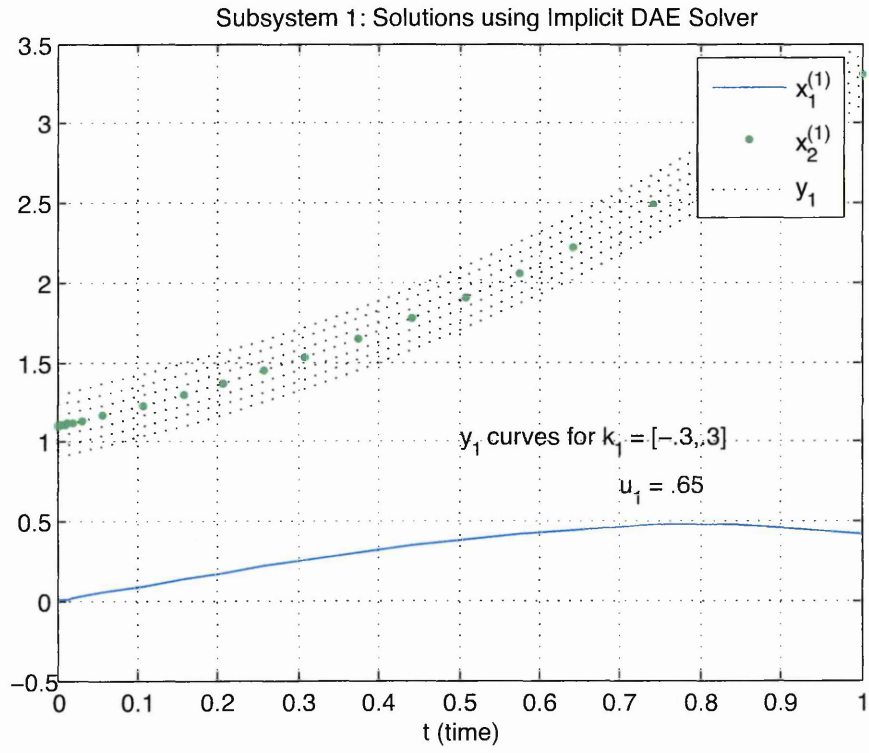


Figure 4.14: Graphs for DAE observer (4.23) (subsystem Σ_1), for input

$u_1 = .65$, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$,

$$y_1(0, u_1) = -k_1 u_1$$

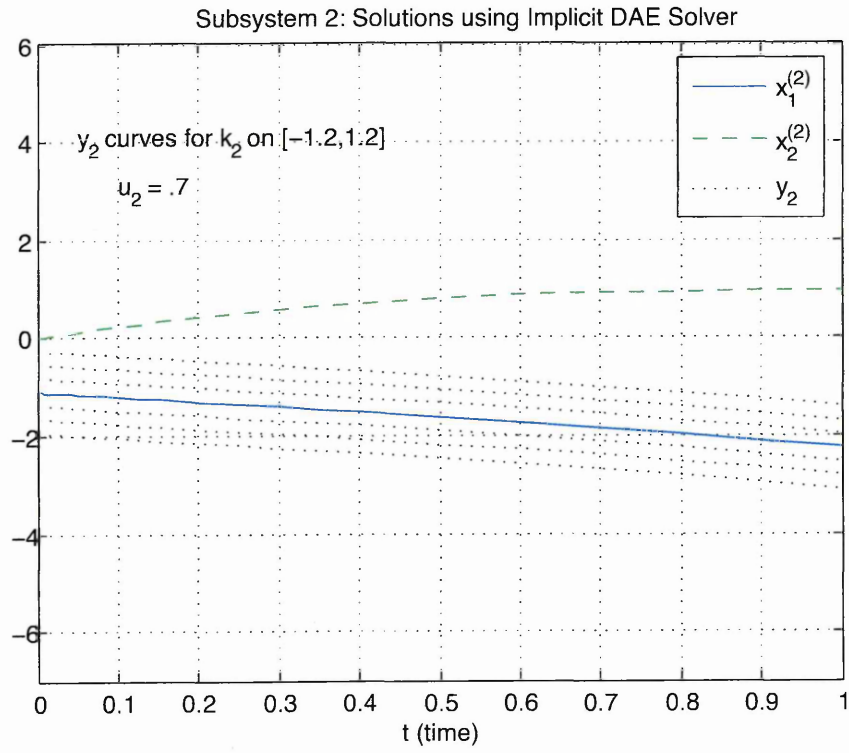


Figure 4.15: Graphs for DAE observer (4.24) (subsystem Σ_2), for input $u_2 =$

.7, parameters $k_2 = [-1.2, 1.2]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$,

$$y_2(0, u_2) = -k_2 u_2$$

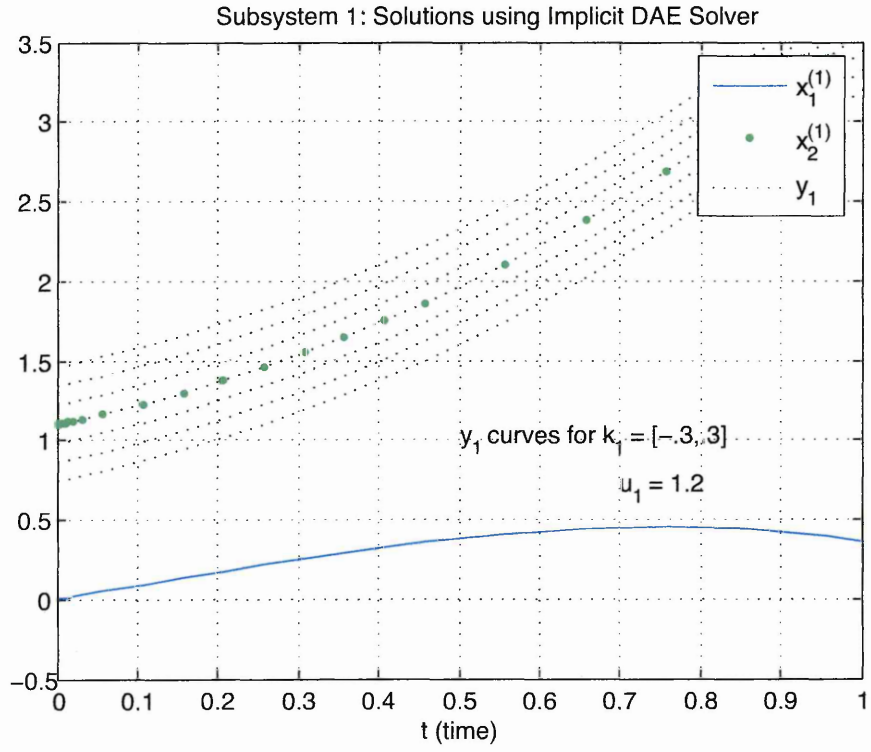


Figure 4.16: Graphs for DAE observer (4.23) (subsystem Σ_1), for input

$u_1 = 1.2$, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$,

$$y_1(0, u_1) = -k_1 u_1$$

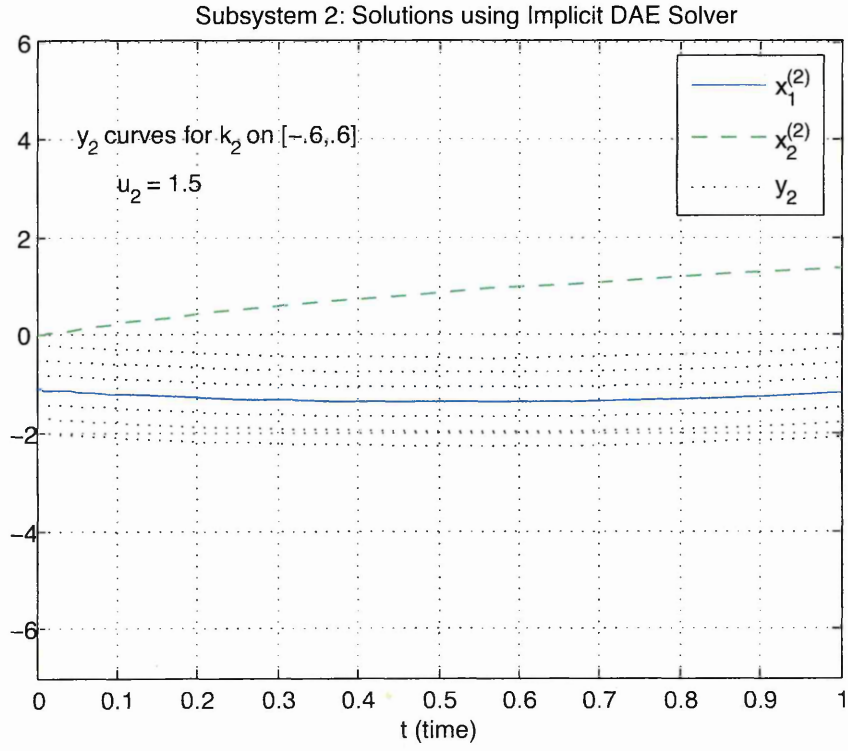


Figure 4.17: Graphs for DAE observer (4.24) (subsystem Σ_2), for input

$u_2 = 1.5$, parameters $k_2 = [-.6, .6]$, initial conditions $x_1(0) = -1.1$, $x_2(0) =$

0 , $y_2(0, u_2) = -k_2 u_2$

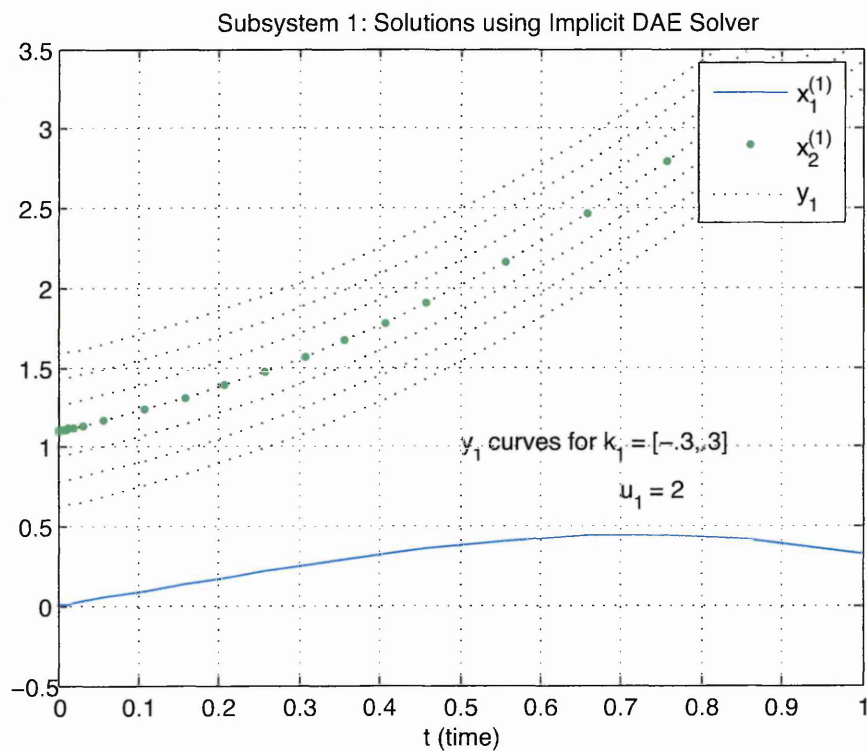


Figure 4.18: Graphs for DAE observer (4.23) (subsystem Σ_1), for input

$u_1 = 2$, parameters $k_1 = [-.3, .3]$, initial conditions $x_1(0) = 0$, $x_2(0) = 1.1$,

$$y_1(0, u_1) = -k_1 u_1$$

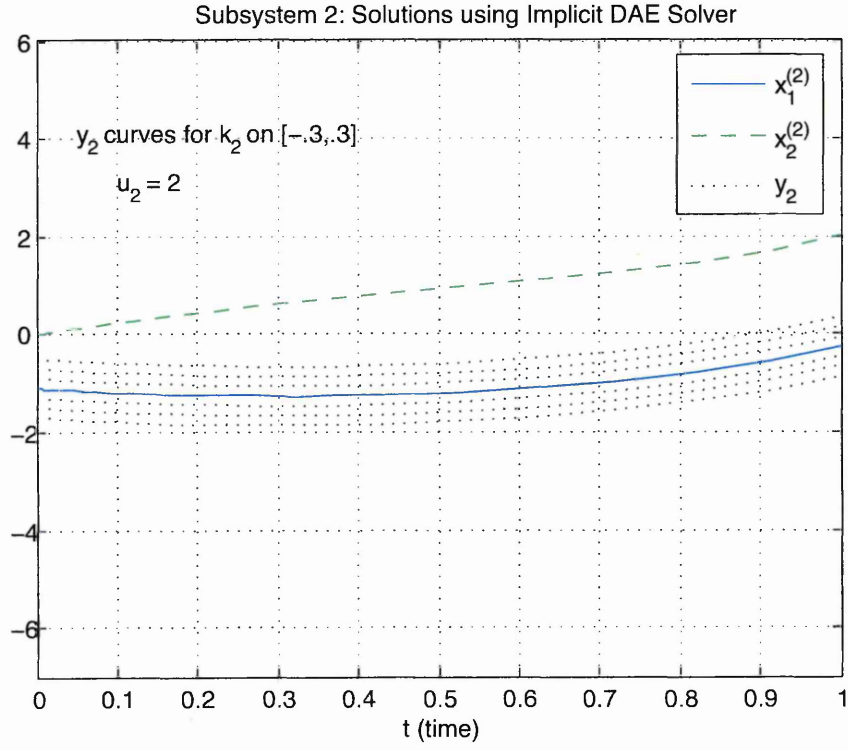


Figure 4.19: Graphs for DAE observer (4.24) (subsystem Σ_2), for input

$u_2 = 2$, parameters $k_2 = [-.3, .3]$, initial conditions $x_1(0) = -1.1$, $x_2(0) = 0$,

$$y_2(0, u_2) = -k_2 u_2$$

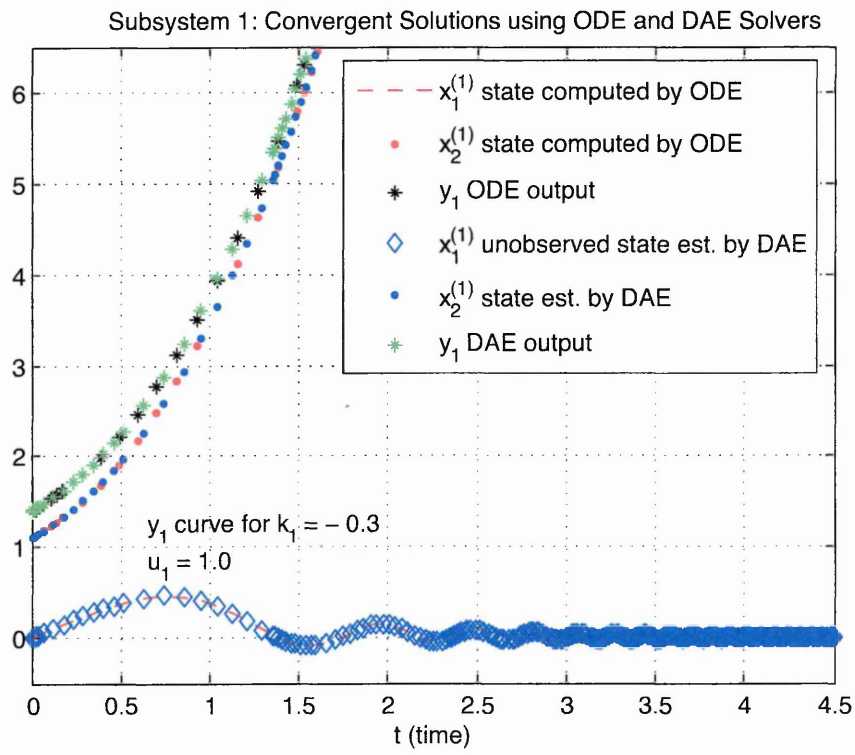


Figure 4.20: Graphs: DAE estimated state variables $\hat{x}_1^{(1)}$, $\hat{x}_2^{(1)}$ and output \hat{y}_1

converge to ODE results.

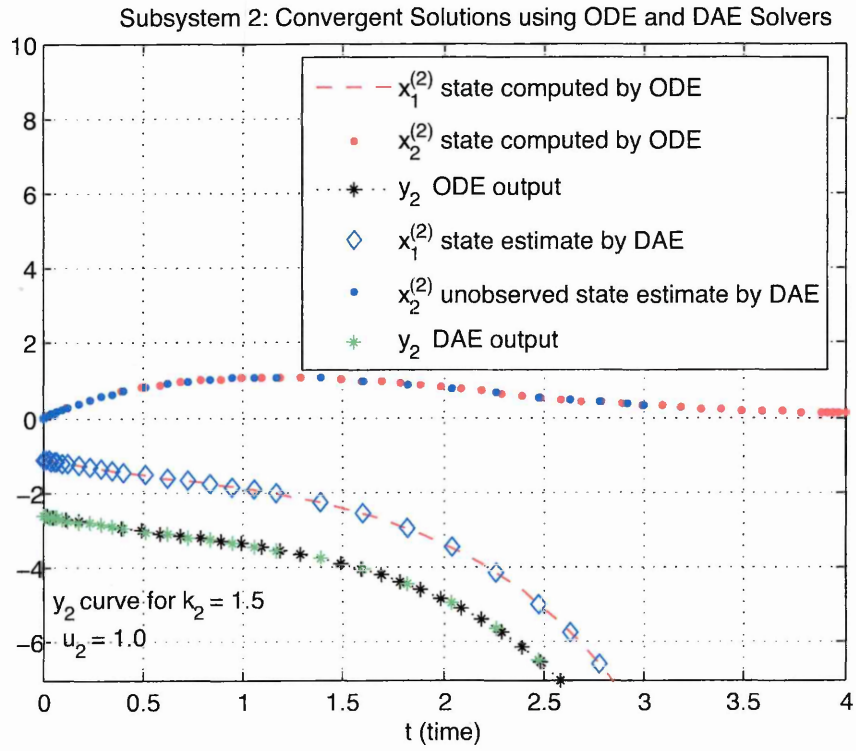


Figure 4.21: Graphs: DAE estimated state variables $\hat{x}_1^{(2)}$, $\hat{x}_2^{(2)}$ and output \hat{y}_2

converge to ODE results.

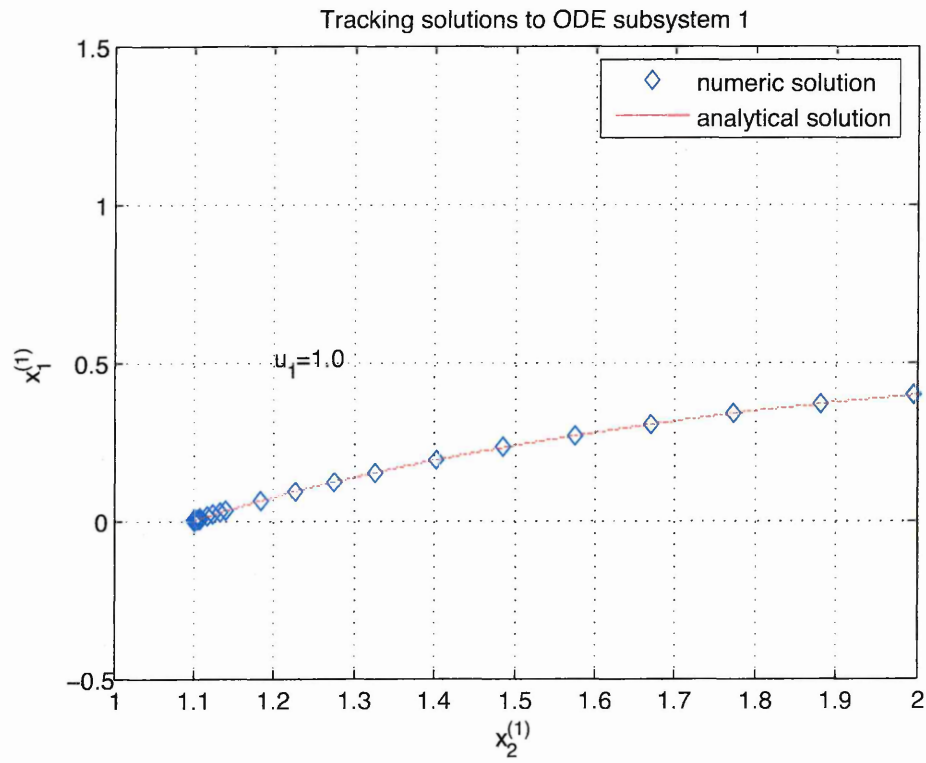


Figure 4.22: Σ_1 Subsystem: Comparing the plot of analytical solution (4.32)

with ODE numerical solution to 4.5, 4.6.

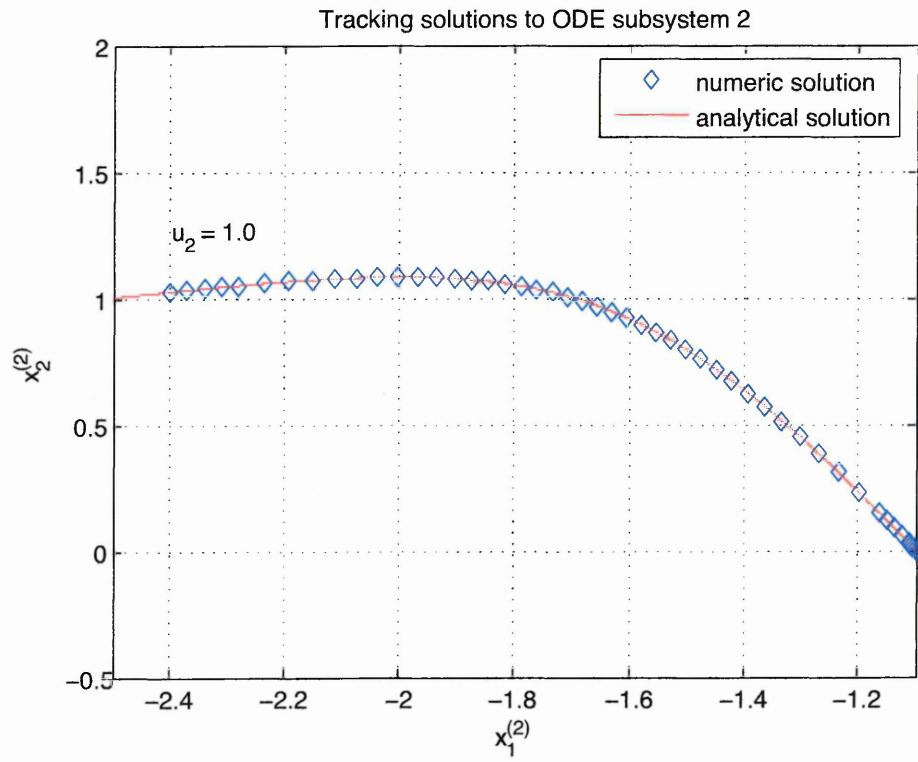


Figure 4.23: \sum_2 Subsystem: Comparing the plot of analytical solution (4.33)

with ODE numerical solution to 4.8, 4.9.

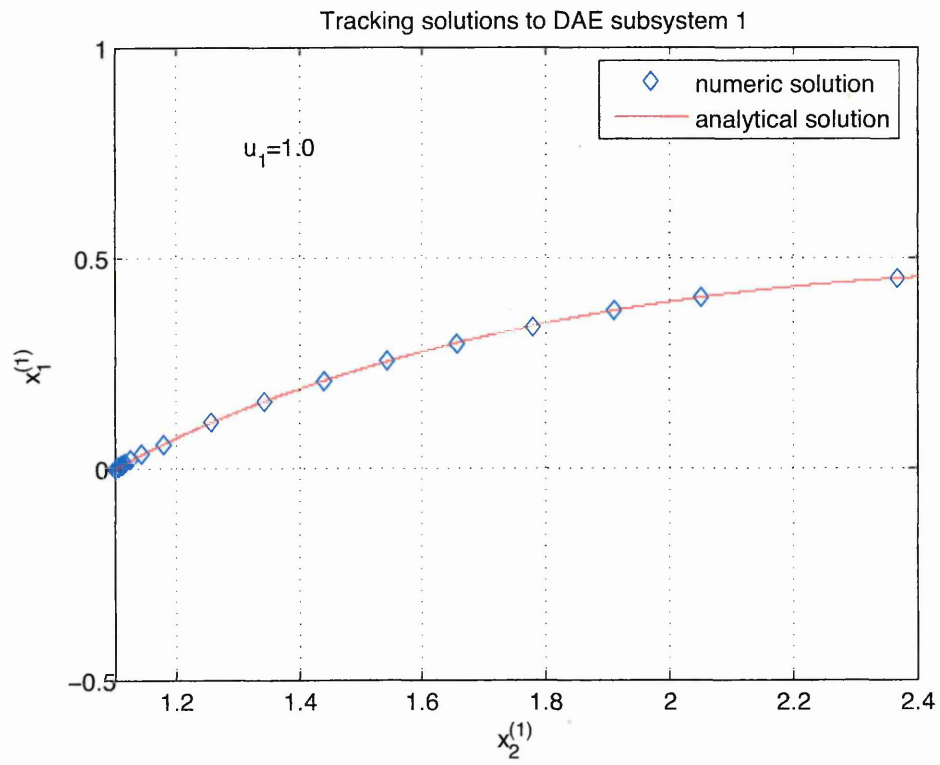


Figure 4.24: \sum_1 Subsystem: Comparing the plot of analytical solution (4.32)

with DAE numerical solution to 4.23.

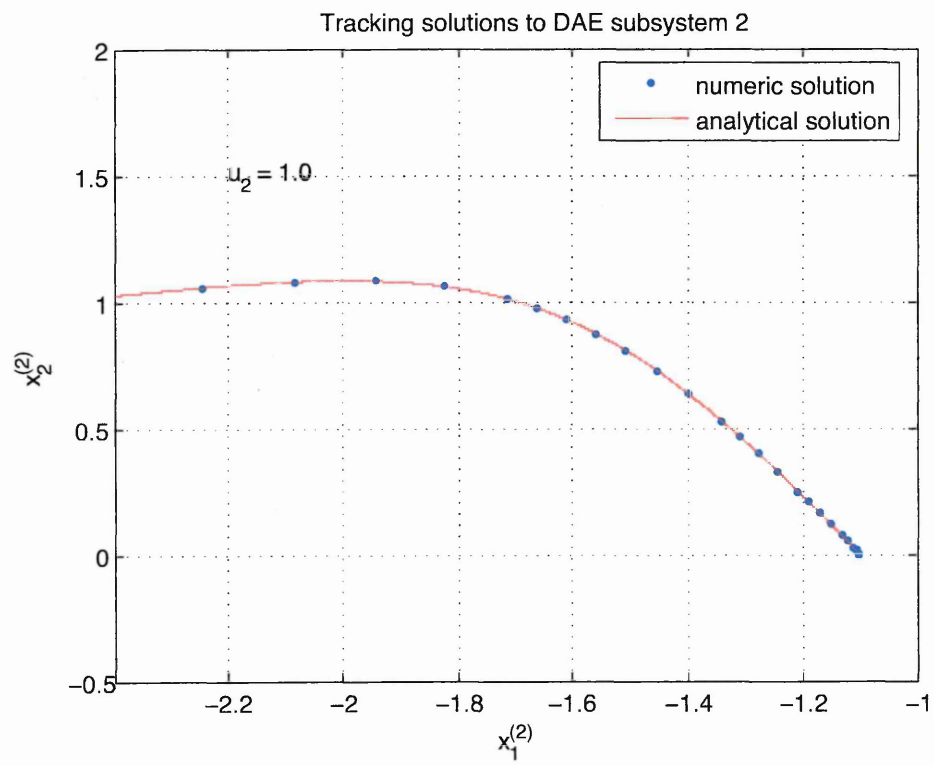


Figure 4.25: Σ_1 Subsystem: Comparing the plot of analytical solution (4.33)

with DAE numerical solution to 4.24.

Chapter 5

Results

In control engineering, observer design problems may be defined as an implicit first order system (see notation as used in beginning of Section 2.2)

$$F(t, y(t), \dot{y}(t)) = 0$$

where F and y are vector valued. We are basically concerned with the study of a system of differential-algebraic equations, or DAE's, where there are algebraic constraints as presented in Sections 2.2, 2.3. Note, it's a DAE, if $\frac{\partial F}{\partial y}$ is singular; however, to convert the DAE into an ODE, it must be nonsingular.

The canonical DAE observer is expressed in the implicit form as presented in Chapter 4, because it's better suited for some numerical integrators available with certain software packages.

In this thesis, the canonical DAE observer was studied and advantages were shown in applying an existing DAE solver for solving this type of problem. An interconnected DAE system where each of the subsystems was expressed as a canonical DAE observer, was considered and a procedure for its implementation using the DAE solver was proposed. We presented a strategy for ensuring input-output stability (IOS) and observability property by interfacing the input-output bound checking capability with a DAE solver (see Section 4.7).

The problem considered was an observer design described by the nonlinear multivariable system

$$\dot{x} = f(x, u)$$

$$y = \bar{h}(x, u)$$

where the state variable is

$$x = (x^{(1)}, x^{(2)})$$

the input is

$$u = (u_1, u_2)$$

the output is

$$y = (y_1, y_2)$$

and the interconnections were placed at the level of inputs and outputs.

The reformulation of the ODE observer system above to a particular type of interconnected canonical DAE observer is demonstrated. In this thesis, the subject of reduced-order observer is discussed in great length to demonstrate the complexity involved when a dynamic system is observable but not all components of the state vector are available. The canonical DAE observer formulation is seen to be closely related to reduced-order observers; however, the estimate of the output state may be computed with greater ease using DAE solvers. It is shown that the full-order observers do not have the sensitivity to high-frequency noise that the reduced-observers have where the estimated state in the case of canonical DAE observers are just as in full-order observers. These advantages are of considerable importance in the implementation of interconnected DAE observer designs. The adaptation of the observer design technique to a system of interconnected DAE systems is discussed. The error analysis on this structure to ensure local stability of the solution was performed and discussed. The gain scheduling and extended linearization techniques are discussed relative to selecting Γ that satisfy local stability criteria and provide a satisfactory design for the interconnecting DAE observers.

The implementation of the interconnected DAE system using existing reliable codes, such as DASSL, is discussed. Solutions to implicit DAEs using

MATLAB & Simulink software is presented. The numerical technique employed by the codes and the advantages of expressing the interconnected subsystems as implicit DAE of uniform index one is discussed. The example given in Sections 4.1, 4.2 is numerically solved using ODE and DAE solvers available with MATLAB & Simulink software. Graphs of solutions for the DAE observer of both the original ODE and the formulated interconnected canonical DAE observer are presented (see Section 4.8).

Numerical integration software such as, DASSL and MATLAB & Simulink, offer a more desirable approach to solving DAE systems, in particular, interconnected DAE observer systems.

A mathematical model of an interconnected DAE system such as the example presented in Chapter 4 can sometimes be a formidable problem to solve analytically. Evidence of the magnitude of the complexity and difficulty can be seen from a publication on Small-Gain Theorem included in Appendix D. However, thanks to the availability of reliable numerical integrators in MATLAB & Simulink and DASSL software as described in a previous section, the dependency on obtaining an analytical solution is minimized. The new method of applying the canonical DAE observers to design and implement observers in the case of constructing interconnected DAE systems has never been approached previously and is discussed at length in Chapter 4. We discuss the Lyapunov method which is especially advantageous in the case of nonlinear dynamic systems. This technique enables one to compute

the necessary parameters which ensure stability of each DAE subsystem and predict with reasonable accuracy the operability conditions required for the stability of the entire interconnected DAE system. This study is presented in Chapter 3. Graphic results presented can attest (see Figures 4.8 to F.2 in Chapter 4) to the usefulness of the above analysis. Section 4.9 contains a derivation of an analytical solution to each subsystem (\sum_i) and presents comparable graphics with those generated by an ODE solver available with the MATLAB & Simulink software package. There is no reason to believe that even though the examples of nonlinear dynamical systems used in Chapter 4 seem almost linear that this is always the case. It is conceivable that there are many nonlinear dynamical systems which could be handled by the new methodology (canonical DAE observer design). Section 1.3 explains that there is no one specific methodology that can serve as the state-of-the-art in observer designing. This thesis establishes the relationship between subsystems by using the DAE approach in order to compute the conditions needed to guarantee the local stability of the system.

This technique offers tremendous potential for designing and implementing DAE observer subsystems of an interconnected system. It minimizes the need for obtaining an analytical solution. However, a scientist or engineer must provide an algorithm to check boundary requirements that must be bound with MATLAB & Simulink or DASSL software. It is strongly believed on my part that this methodology should have a high impact in the area of

constructing observers in the future.

To the best of the writer's knowledge, there exist no publications or known attempts by the research community relative to the problem of applying the new approach to output feedback stability studies, especially, to the case of either the interconnected DAE observer or full-order observer systems (see system 1.6, 1.7). As mentioned in Section 1.3, [25] should be referenced.

Considerations for Future Research:

In this thesis, a nonlinear observer design system is considered in affine form (discussed in Section 1.3) as follows:

$$\dot{x} = \bar{f}(x) + g(x)u$$

$$y = h(x) + \bar{k}(x)u$$

where $\bar{k}(x)$ is considered as a nonzero constant. A possible extension to the work presented is to consider when $\bar{k}(x)$ is not a constant but a function of an independent vector x i.e. $\bar{k} : \hat{D} \rightarrow \mathbb{R}$, $\hat{D} \subset \mathbb{R}^n$ as a domain. This problem should be considered in the DAE framework. That is, $h(\hat{x})$, $h_x(\hat{x})$ should be replaced by $h(\hat{x}, u)$, $h_x(\hat{x}, u)$, respectively, in equations (2.5) and (2.6), (2.7), where $[h_x(\hat{x}, u), h_x(\hat{x}, u)^t]$ is positive definite and DAE system (2.5) is index one. Also, the construction of $\Gamma(\hat{x}, u)$ (see(3.19)) should still be Hurwitz with respect to the same replacements.

The formulation of a canonical DAE observer is based on a DAE observer of index 1 as shown in Section 2.3. Perhaps, a study of DAE observers of higher index than one leading to a canonical DAE observer should be considered.

Another interesting project to consider is the numerical integration implementation aspect of the interconnected DAE observer (4.28), using the DAE solver, DASSL. The example given in Section 4.1 may be useful in gaining experience in solving, numerically, an interconnected DAE observer problem.

Appendix A

Optimal Control

A.1 Problem Statement

The purpose of this appendix is to demonstrate the application of the technique developed in the paper [16] to the minimization of an optimal control problem. The problem is stated in its original form (see (A.1) below) and con-

verted in terms of a DAE system. The DAE system is a general formulation obtained by using calculus of variation to get the Kuhn-Tucker conditions and the complementarity conditions are expressed as equalities with the addition of a new variable. The general transformation process is given in [16] and is not repeated here; however, the solution will be computed.

Let us consider a control problem in which it is desired to maintain a minimum functional, $J(u)$; that is,

minimize :

$$J(u) = \int_0^1 (x + u)dt \quad (\text{A.1})$$

where x, u are functions of time t .

Subject to the constraints :

$$\frac{dx}{dt} = x - u, \quad x(0) = 5 \quad (\text{A.2})$$

and the inequality constraint

$$\frac{1}{2} \leq u \leq 1 \quad (\text{A.3})$$

or

$$0 \leq u - \frac{1}{2}$$

$$0 \leq 1 - u$$

It is desired to find functions x and u which minimize $J(u)$, given x and u related by differential equation (A.2) and boundary condition (A.3).

A.2 Conversion to General Statement

Using the procedure as outlined in [16], equation (12), yields the following DAE system without inequality:

$$\frac{dx}{dt} = x - u, \quad x(0) = 5 \quad (\text{A.4})$$

$$\frac{dv}{dt} = -(v + 1), \quad v(1) = 0 \quad (\text{A.5})$$

$$v + p_1^- - p_2^- - 1 = 0 \quad (\text{A.6})$$

$$p_1^+ - u + \frac{1}{2} = 0$$

$$p_2^+ + u - 1 = 0$$

where the inequalities were eliminated using, $p = g - w$, $g = \max(0, p) = p^+$, $w = \max(-p, 0) = p^-$.

Thus, we have a system with 5 variables (x, u, v, p_1, p_2) which must satisfy 2 differential equations and 3 algebraic equations but no inequalities. Included are 2 boundary conditions for the differential equations. The DAE system above is index 1, where the algebraic variables do not appear in the form of derivatives. Note, p_1 , p_2 and u can be expressed explicitly in terms of differential variables.

A.3 Solution

If $p_1 < 0$ or $p_2 < 0$ the system of DAE is such that the algebraic variables, which do not appear in the form of derivatives, in this case p_1 , p_2 and u , can be explicitly expressed in terms of the differential variables, and so the system is of index one.

These lead to the consideration of 4 cases to be considered:

- 1) $p_1 < 0$, $p_2 < 0$ yields $u = \frac{1}{2}$ and $u = 1$, which is impossible.
- 2) $p_1 > 0$, $p_2 > 0$ yields for $u > \frac{1}{2}$ and $u < 1$, $v = 1$, which is of index > 1 . But $dv/dt = 0$, $-(v + 1) = -2$, and this case is impossible.
- 3) $p_1 > 0$, $p_2 < 0$ yields for $u > \frac{1}{2}$ and $u = 1$, $v > 1$
- 4) $p_1 < 0$, $p_2 > 0$ yields for $u = \frac{1}{2}$ and $u < 1$, $v < 1$

In order that the solution for $x(t)$, $v(t)$ be consistent, case 1) and case 2) are discarded since they are impossible and consider case 3) and case 4).

Since the DAE system is index 1, we can solve for v as shown. Solving differential equation (A.5) and applying the boundary condition, we obtain

$$v(t) = e^{1-t} - 1 \tag{A.7}$$

where $v(t) = 1$ yields $t = 1 - \ln 2 \approx .307$, $v(t) > 1$ and $u = 1$ for $t < .307$, and

$v(t) < 1$ and $u = \frac{1}{2}$ for $t > .307$.

Solving differential equation (A.4), with $x(0) = 5$, $u = 1$ for $t < .307$ and $u = \frac{1}{2}$ for $t > .307$, the following is obtained:

$$x(t) = \begin{cases} 4e^t + 1 & 0 \leq t \leq .307 \\ 4e^t + e^{t-1} + \frac{1}{2} & t \geq .307 \end{cases} \quad (\text{A.8})$$

Now the optimal value of the objective function is computed in the following way:

$$J(u) = \int_0^{.307} (x + 1)dt + \int_{.307}^1 (x + \frac{1}{2})dt \quad (\text{A.9})$$

$$= \int_0^{.307} (4e^t + 2)dt + \int_{.307}^1 (4e^t + e^{t-1} + 1)dt \quad (\text{A.10})$$

After integrating and computing, we arrive at

$$J(u) = 8.679$$

A couple of checks is applied to ensure that the answer is plausible.

Consider $u = 1$ throughout the entire range of t . Then,

$$J(u) = \int_0^1 (4e^t + 2)dt = 8.872$$

which is greater than that obtained. Similarly, suppose $u = \frac{1}{2}$ for all $t \in (0, 1)$.

Then,

$$x(t) = 4.5e^t + 0.5, \quad J(u) = 9.731$$

which is again greater. Hence, the answer is acceptable.

A.4 Tractability Index

Here we demonstrate how one determines the index of a DAE system using the tractability index concept described in [16].

The DAE system is given by equations (A.4) - (A.6). Following the same notation as used in [16] for the matrix chains and expressing the vector of dependent variables by $\underline{x}(x, v, u, p_1, p_2)$, the B matrix becomes for case 3) ($p_1 > 0, p_2 < 0$),

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Also, we have

$$A = \begin{bmatrix} I_2 & \\ & O_3 \end{bmatrix}, Q = \begin{bmatrix} O_2 & \\ & I_3 \end{bmatrix}$$

The following equations must be computed in order to determine the index of the system:

$$\begin{aligned} A_0 &= A \\ Q_0 &= Q \\ P_0 &= I - Q_0 \\ B_0 &= B - A_0 P_0' \\ A_1 &= A_0 + B_0 Q_0 \end{aligned}$$

Then,

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The determinant of A_1 , $\det(A_1) = -1$, which means the system is index 1.

Similarly, for case 4) ($p_1 < 0, p_2 > 0$),

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Again, the determinant of A_1 , $\det(A_1) = -1$.

Appendix B

Laplace Transformation

B.1 Partial List of Laplace Transform Pairs

$f(t)$	$F(s)$
unit impulse $\delta(t)$	1
unit step $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{a^2(at-1+e^{-at})}$	$\frac{1}{s^2(s+a)}$
$\frac{\sin \omega t}{\omega}$	$\frac{1}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\frac{\sinh \omega t}{\omega}$	$\frac{1}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$

B.2 Properties of Laplace Transforms

multiplication by constant	$\mathcal{L} [Af(t)] = AF(s)$
linearity	$\mathcal{L} [f_1(t) + f_2(t)] = F_1(s) + F_2(s)$
first derivative	$\mathcal{L} [\frac{d}{dt}f(t)] = sF(s) - f(0)$
multiplication by exponential	$f(t)e^{-at} = F(s+a)$
multiplication by t	$tf(t) = \frac{-dF(s)}{ds}$
time translation	$\mathcal{L} [f(t-a)1(t-a)] = e^{-as}F(s)$
time integration	$\mathcal{L} [\frac{1}{t}f(t)] = \int_s^\infty F(s)ds$
scale change	$\mathcal{L} [f(\frac{t}{a})] = aF(as)$
initial value	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$
final value	$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

B.3 Example

To demonstrate an alternative approach to solving differential equations, the Laplace transform will be applied to differential equation given by (1.40),

$$\dot{\hat{\xi}}(t) = -5\hat{\xi}(t) - 20(3 - 2e^{-t} - t) - 1.$$

For t , we can obtain $\mathcal{L}[t]$ using the definition $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$, where $f(t) = t$. Upon integration, $\frac{1}{s^2}$ is obtained. However, with the tables given in B.1 and B.2, we can compute the the following transform:

$$\hat{\xi}(s) - \hat{\xi}(0) + 5\hat{\xi}(s) = -20\left[\frac{3}{s} - \frac{2}{s+1} - \frac{1}{s^2}\right] - \frac{1}{s}$$

Assuming, $\hat{\xi}(0) = -4$, we obtain the following:

$$\hat{\xi}(s) = -\frac{4}{s+5} - 20\left[\frac{3}{s(s+5)} - \frac{2}{(s+1)(s+5)} - \frac{1}{s^2(s+5)}\right] - \frac{1}{s(s+5)}$$

Using the Table in B.1, we obtain the inverse transform $\mathcal{L}^{-1}[\hat{\xi}(s)]$:

$$\hat{\xi}(t) = -4e^{-5t} - 20\left[\frac{3}{5}(1 - e^{-5t}) - \frac{1}{2}(e^{-t} - e^{-5t}) - \frac{1}{25}(5t - 1 + e^{-5t})\right] - \frac{1}{5}(1 - e^{-5t})$$

Hence,

$$\hat{\xi}(t) = 10e^{-t} + 4t - 13 - e^{-5t}$$

Appendix C

Observers

C.1 Reduced-Order Observers

In this appendix, Luenberger's theory of reduced-order observers for continuous, deterministic dynamical systems is presented (see example in Section 1.6). The full state vector, x , is assumed not available. Since only the vector,

y , which is linearly related to x is available for the purpose of controlling the system, one wants to find an estimate, \hat{x} , of the full state vector x . We consider an n^{th} order system as described by (1.3), (1.4) in Section 1.3, where $u(t)$ is a control input, and $y(t)$ are observations of available states. C (only constant coefficient matrices are considered) is referred to as a measurement matrix ($p < n$) which is assumed to be of full rank p . Thus, $y(t)$ represents p linearly independent combinations of the state vector, $x(t)$. It is also assumed that the system described above is observable as defined in many linear systems texts (refer to Section 1.3).

Reduced-order observers eliminate redundancy which occurs when an observer constructs an estimate of the entire state. Part of the state as given by the system output is already available by direct measurement. The notion of reduced-order observers is that if some components of $x(t)$ are measurable but others are not, then the objective is to determine only the missing components using estimates. The following demonstrates the computation of a reduced-order observer (see [7]).

Consider the n^{th} order system as described by (1.3), (1.4), where u is a deterministic (control) input. The observations of the state are available according to (1.4). It is desired to provide an estimate $\hat{x}(t)$ of the state using an $(n - p)^{th}$ order observer (see Section 1.4). This is done by introducing an

$(n - p)$ dimension transformation vector

$$\xi(t) = Tx(t) \quad (C.1)$$

such that

$$\begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} T \\ C \end{bmatrix} x(t)$$

where

$$\begin{bmatrix} T \\ C \end{bmatrix} \quad (C.2)$$

is a nonsingular matrix. The partition matrix assumes C has rank p . T has dimension $(n - p) \times n$, and C has dimension $p \times n$. The vector $\xi(t)$ represents $(n - p)$ linear combinations of the system states which are linearly independent of the measurements, $y(t)$. It is therefore possible to obtain the inverse transformation

$$x(t) = \begin{bmatrix} T \\ C \end{bmatrix}^{-1} \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} \quad (C.3)$$

For convenience, define

$$\begin{bmatrix} T \\ C \end{bmatrix}^{-1} \stackrel{\text{def}}{=} [L \mid M] \quad (C.4)$$

so that

$$x(t) = L\xi(t) + My(t) \quad (C.5)$$

where L is of dimension $n \times (n - p)$ and M is of dimension $n \times p$. Note, matrices T, C, L and M are assumed constant.

The concept of observers is based on devising an $(n-p)^{th}$ order estimate $\hat{\xi}(t)$ (estimator) for the transformed state vector $\xi(t)$, which can then be used to reconstruct an estimate of the original vector $x(t)$, according to the relationship of equation (C.5).

In the following development, the form of the dynamic observer is presented and the corresponding error equations derived.

Now, some constraint relationships are established between the L, M, T and C matrices.

Using equation (C.4) above, the following equations are obtained:

$$LT + MC = I \quad (C.6)$$

and

$$\begin{bmatrix} T \\ - \\ C \end{bmatrix} [L \mid M] = \begin{bmatrix} TL & TM \\ CL & CM \end{bmatrix} = I \quad (C.7)$$

Note, $TL = I_{n-p}$, $CM = I_p$, $TM = \underline{0}$ and $CL = \underline{0}$. Also, $I = I_n$.

A differential equation for $\xi(t)$ can be obtained from the above appropriate equations (1.3), (C.1), (C.5). The result is

$$\dot{\xi} = (TAL)\xi + (TAM)y + TBu \quad (C.8)$$

or

$$\dot{\xi} = TA(L\xi + My) + TBu \quad (\text{C.9})$$

We are therefore led to an observer of the form

$$\dot{\hat{\xi}} = (TAL)\hat{\xi} + (TAM)y + TBu \quad (\text{C.10})$$

where

$$\hat{x} = L\hat{\xi} + My \quad (\text{C.11})$$

We know that for every initial state of the system, $x(t_0)$, there exists an initial state $\hat{\xi}(t_0)$ of the observer given by (C.10), (C.11) such that $\hat{x}(t) = x(t)$ for any $u(t)$, for all $t \geq t_0$. However, the initial condition may not be known for x and ξ , so it is appropriate to consider the propagation of the observer error $\tilde{\xi}$. The observers exhibit the property that the observer error, defined by $\tilde{\xi} = \hat{\xi} - \xi$ decays exponentially to zero (see discussion at end of Section 1.3). From (C.10) and (C.9), the following differential equation is obtained

$$\dot{\tilde{\xi}} = (TAL)\tilde{\xi} \quad (\text{C.12})$$

If the eigenvalues of matrix (TAL) are chosen properly by appropriate specification of T , L and M , subject to the constraints of (C.6) and (C.7), then the stability of the observer and the behavior of $\tilde{\xi}$ can be determined. For example, since we consider the system (C.12) as linear time-invariant, then if the eigenvalues of TAL have negative real parts, the observer is stable (also see Sections 1.3, 1.4). However, in this example a reduced-order observer is

constructed where $n - p = 1$ and has an eigenvalue with a negative real part (see C.17 and its discussion). That is, if (C.12), a linear invariant dynamical equation is unobservable, the order can be reduced such that the reduced equation still has the same initial state as the original dynamical system.

Now the differential equation for \tilde{x} can be derived. From equations (C.5) and (C.11), the relationship for $\tilde{\xi}$ is given by

$$\tilde{x} = L\tilde{\xi}$$

Owing to the convergence properties of $\tilde{\xi}$ as discussed previously, \tilde{x} tends uniformly and asymptotically to zero. The corresponding relationship $\tilde{\xi} = T\tilde{x}$ can also be obtained by $T\tilde{x} = TL\tilde{\xi}$ and invoking the constraint $TL = I$ (see (C.7)).

Using these relationships, the differential equation for \tilde{x} is

$$\dot{\tilde{x}} = L\dot{\tilde{\xi}} \tag{C.13}$$

After substituting (C.12) for $\tilde{\xi}$ in the above equation, we obtain

$$\dot{\tilde{x}} = (LTAL)\tilde{\xi}$$

This simplifies even further to

$$\dot{\tilde{x}} = LTA\tilde{x} \tag{C.14}$$

Hence, after substituting $I - MC$ for LT in (C.14), we finally arrive at the following error estimation:

$$\dot{\tilde{x}} = (A - MCA)\tilde{x} \quad (\text{C.15})$$

Note, from the given system (1.3) and (1.4) where A and C are described, we see from (C.15) that the error estimation depends only on choice of M so that $\tilde{x} \rightarrow 0$. That is, if the system is completely observable, the matrix M can be chosen to achieve any desired set of $(n - p)$ eigenvalues for error response. For the linear time-invariant system, we can choose an arbitrary eigenvalue for A and pick M such that $(A - MCA)$ has nonzero eigenvalues, then choose L, T consistent with $LT + MC = I$. Note, the choice of L and T which satisfies $LT + MC = I$ is not unique.

Clarification: Suppose for a time-invariant system, we choose an arbitrary set of eigenvalues, Λ (complex conjugate pairs), pick M such that $A - MCA$ has Λ as its nonzero set of eigenvalues, and choose L and T consistent with $LT + MC = I$, where the choice of L and T is not unique. Now suppose that an allowable pair of matrices (L^*, T^*) is chosen. Then the pair (L, T) given by $L = L^* M^{-1}, T = MT^*$ also satisfies $LT + MC = I$, where M is any nonsingular matrix. The set of all allowable pairs (L, T) defines an equivalent class of observers which exhibit the same error behavior.

In the following discussion, the construction of a reduced-order observer using the above methodology is demonstrated for the example given in Section 1.6.

From equation (C.7), $CM = I$, so that M is constrained to be of the form

$$M = \begin{bmatrix} 1 \\ b \end{bmatrix} \quad (\text{C.16})$$

Then the matrix in the equation of the error dynamics (C.15) becomes

$$A - MCA = \begin{bmatrix} 0 & 0 \\ 0 & -(\beta + b) \end{bmatrix} \quad (\text{C.17})$$

Since $n - p = 1$, the reduced-order observer is of order one and is specified by the single eigenvalue

$$\mu = -(\beta + b) \quad (\text{C.18})$$

We can choose μ so that

$$\mu = -5\beta \quad (\text{C.19})$$

where $\beta \neq 0$,

which implies $b = 4\beta$. That is, since L and T must satisfy the constraints $LT + MC = I$, $CL = 0$, $TL = I$, $CM = I$ and $TM = 0$, then a possible choice of L and T satisfying (C.6) and (C.7) is

$$L = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\text{C.20})$$

and

$$T = \begin{bmatrix} -4\beta & 1 \end{bmatrix} \quad (\text{C.21})$$

C , M are given respectively, by (1.18) and (1.19) Section 1.6. Also, $TAL = -5\beta < 0$. This completes the specification of the first order observer.

From (C.11), we have

$$\begin{aligned}\hat{x}_1(t) &= y(t) \\ \hat{x}_2(t) &= \hat{\xi}(t) + 4\beta\hat{x}_1(t)\end{aligned}$$

from the coefficient matrices.

The system of differential equations represented by Figure 1.3 is given directly by equations (1.14) - (1.19) in Section 1.6. Using the given parameters and initial conditions, the solution is obtained as follows:

$$x_1(t) = 3 - 2e^{-t} - t \quad (\text{C.22})$$

$$x_2(t) = 2e^{-t} - 1 \quad (\text{C.23})$$

Similarly for Figure 1.4, using (C.9), (C.10), (C.11), (1.18), (1.19), (C.16), (C.21), as follows:

From (C.11), we have

$$\hat{x}_1 = y \text{ and } \hat{x}_2 = \hat{\xi} + 4\beta y \text{ or } \hat{\xi} = \hat{x}_2 - 4\beta y.$$

Then, from equation (C.10) in the time-invariant case for this example,

$$\dot{\hat{\xi}} = -5\beta\hat{\xi} - 5\beta by + lu$$

since $\dot{L} = \dot{M} = 0$, $TA = [0 \quad -5\beta]$, and $TB = l$. From the given parameters and initial values, the following equations are determined:

Because $\hat{x}_1(t) = x_1(t)$, we have

$$\hat{x}_1(t) = 3 - 2e^{-t} - t \quad (\text{C.24})$$

Substituting (C.24) and $\beta = 1$ in $\hat{x}_2 = \hat{\xi} + 4\beta\hat{x}_1$, we obtain

$$\hat{x}_2(t) = 12 - 8e^{-t} - 4t + \hat{\xi} \quad (\text{C.25})$$

Then, from (C.10), we have $\dot{\hat{\xi}} = -5\hat{\xi} - 20\hat{x}_1 - 1$ which, after substituting for \hat{x}_1 , yields

$$\dot{\hat{\xi}}(t) + 5\hat{\xi} = -20(3 - 2e^{-t} - t) - 1 \quad (\text{C.26})$$

Integrating (C.26), yields

$$\hat{\xi}(t) = 10e^{-t} + 4t - 13 - e^{-5t} \quad (\text{C.27})$$

$$\hat{x}_2(t) = 2e^{-t} - e^{-5t} - 1$$

Note, the above equations can be compared with those obtained Section 1.6 on Reduced-Order Observers.

Appendix D

Small-Gain Theorem

D.1 Definitions and Theorem

Since the interconnected system here is based on the control system given by

$$\dot{x} = f(x, u) \tag{D.1}$$

$$y = \bar{h}(x, u)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and output $y \in \mathbb{R}^p$, where f and h are smooth functions, we consider this system. Some definitions and notations from [18] are introduced in order to present the concept of small-gain theorem relative to interconnected systems. The definitions given below are restricted as to their relevancy, as far as this thesis is concerned, in demonstrating the advantage of applying the DAE solver to our case of interconnected canonical DAE systems.

A function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be proper if $V(x)$ tends to $+\infty$ as $|x| \rightarrow +\infty$. A proper function is called radially unbounded in the automatic control literature.

A function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class K, if it is continuous, increasing, and is zero at zero. It is of class K_∞ if, in addition, it is proper.

For any γ of K_∞ , the inverse γ^{-1} is well defined and is again of class K_∞ .

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class KL if, for each fixed t , the function $\beta(\cdot, t)$ is of class K and, for each fixed s , the function $\beta(s, \cdot)$ is non-increasing monotonically and tends to zero at infinity.

$|\cdot|$ stands for Euclidean norm.

Id denotes identity function.

The following is some general background information from Lebesgue Theory leading to the ensuing definition for essential supremum (ess. sup):

A real number M is said to be an essential bound for a function f whenever

$|f(t)| \leq M$ for almost all t .

A function is called essentially bounded if it has an essential bound. Therefore, a function is essentially bounded if it is bounded except possibly on a set of measure zero. The essential supremum (ess sup) of a function is defined by $\|f\|_\infty = \inf \{M \geq 0 : |f(t)| \leq M \text{ holds for almost all } t\}$. If f does not have any essential bound, then it is understood that $\|f\|_\infty = \infty$. Note that $|f(t)| \leq \|f\|_\infty$ holds for almost all t .

For any measurable function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$,

$\|u\|$ denotes ess. sup. $\{|u(t)|, t \geq 0\}$ and, for any pair of times $0 \leq t_1 \leq t_2$, the truncation is defined as follows:

$$u_{[t_1, t_2]} = \begin{cases} u(t) & \text{if } t \in [t_1, t_2], \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $u_{[0, T]}$ is the usual truncated function and to simplify the notation

we let

$$u_T = u_{[0, T]}$$

Definition 1: System (D.1) is said to have the unboundedness observability (UO) property if a function α^0 of class K and a nonnegative constant D^0 exist such that, for each measurable essentially bounded control $u(t)$ on $[0, T)$ with $0 < T \leq \infty$, the solution $x(t)$ of (D.1) right maximally defined

on $[0, T')$ ($0 < T' \leq T$) satisfies

$$|x(t)| \leq \alpha^0(|x(0)| + \|(u_t^T, y_t^T)^T\|) + D^0, \quad \forall t \in [0, T') \quad (\text{D.2})$$

Clarification: Let $\|u\|_\infty = \inf \{M \geq 0 : |u(t)| \leq M \text{ holds for almost all } t \in [0, T], 0 \leq T \leq \infty\}$.

Then the solution of $x(t)$ of (D.1) must be defined up to T' where $[0, T')$ ($0 < T' \leq T < \infty$). That is, T' is the furthest right of the interval where $x(t) < \infty$.

Definition 2: System (D.1) is input-output stable (IOS) if a function β of class KL, and a function γ of class K, called a (nonlinear) gain from input to output, and a nonnegative constant d exist such that, for each initial condition $x(0)$, each measurable essentially bounded control $u(\cdot)$ on $[0, \infty)$ and each t in the right maximal interval of definition of the corresponding solution of (D.1), we have

$$|y(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|) + d \quad (\text{D.3})$$

But, here $d = 0$.

Consider the interconnected system (4.20) where the functions f_1, f_2, \bar{h}_1 , and \bar{h}_2 are smooth and a smooth function \bar{h} exists such that $(y_1, y_2) = \bar{h}(x_1, x_2, u_1, u_2)$ is the unique solution of $y_i = \bar{h}_i(x_i, u_i)$ for $i = 1, 2$.

Now modifying “The Small-Gain Theorem” from [18], the adapted statement becomes the following theorem:

Theorem:

Suppose (4.20) is IOS with respect to input (y_2, u_1) , (y_1, u_2) , and output y_1, y_2 . Also $(\beta_1, (\gamma_1^y, \gamma_1^u), d_1)$, and $(\beta_2, (\gamma_2^y, \gamma_2^u), d_2)$ are two triples satisfying (D.3) with $d_i = 0$ and $-y_2 = u_1$ and $y_1 = u_2$, namely

$$|y_1(t)| \leq \beta_1(|x_1(0)|, t) + \gamma_1^y(\|y_{2t}\|) + \gamma_1^u(\|u_1\|) + d_1 \quad (\text{D.4})$$

$$|y_2(t)| \leq \beta_2(|x_2(0)|, t) + \gamma_2^y(\|y_{1t}\|) + \gamma_2^u(\|u_2\|) + d_2$$

Also, suppose (4.20) has the UO property with couple (α_i^0, D_i^0) ($i = 1, 2$). If two functions ρ_1 and ρ_2 of class K_∞ and a nonnegative real number s , $\forall s \geq 0$ satisfying

$$(Id + \rho_2) \circ \gamma_2^y \circ (Id + \rho_1) \circ \gamma_1^y(s) \leq s \quad (\text{D.5})$$

$$(Id + \rho_1) \circ \gamma_1^y \circ (Id + \rho_2) \circ \gamma_2^y(s) \leq s$$

exist, then system (4.20) with $u = (u_1, u_2)$ as input, $y = (y_1, y_2)$ as output, $x = (x_1, x_2)$ as state is IOS with the property UO (with $D^0 = 0$ when $d_i = D_i^0 = 0, (i = 1, 2)$). More specifically, for each pair of class K_∞ , the functions (r_3, ρ_3) , and a function β of class KL exist such that system (4.20) is IOS with the triple $(\beta, r_1 + r_2 + r_3, 0)$, where

$$r_1(s) = (Id + \rho_1^{-1}) \circ (Id + \rho_3)^2 \circ [\gamma_1^u + \gamma_1^y \circ (Id + \rho_2^{-1}) \circ (Id + \rho_3)^2 \circ \gamma_2^u](s)$$

$$r_2(s) = (Id + \rho_2^{-1}) \circ (Id + \rho_3)^2 \circ [\gamma_2^u + \gamma_2^y \circ (Id + \rho_1^{-1}) \circ (Id + \rho_3)^2 \circ \gamma_1^u](s)$$

◇

Note, 'o' is the mathematical notation for composition of functions.

Definitions 3:

Weak triangular inequality -

For any function γ of class K , any function of ρ of class K_∞ such that $\rho - Id$ is of class K_∞ , any nonnegative real numbers a and b we have:

$$\gamma(a + b) \leq \gamma(\rho(a)) + \gamma(\rho \circ (\rho - Id)^{-1}(b))$$

This inequality generalizes another and is established by remarking ([18]) that for any function σ of class K_∞ , we have

$$\gamma(a + b) \leq \max_{0 \leq s \leq \sigma(a)} \{\gamma(a + s)\} + \max_{0 \leq s \leq \sigma^{-1}(b)} \{\gamma(s + b)\}$$

The proof of the Small-Gain Theorem is quite lengthy and is presented in [18]. We present the theorem here to demonstrate its complexity from an analytical perspective so as to illustrate that the operations of the bounded functions must be valid on proper time intervals in order that the results make sense.

Condition (D.3) has been introduced in [31] to state an IOS result in the operator setting without making the role of initial conditions explicit. This condition is a non-linear version of the classical small-gain condition (see [19]). Sufficient conditions to check condition (D.3) are given in [31].

Note that the initial conditions were taken into account and the gain function γ of the closed-loop system is in terms of the gains of two subsystems.

Appendix E

List of Symbols & Notation

\equiv	identically equal ¹
$O(\cdot)$	order of magnitude
x	state vector
y	output vector
u	input vector
$f(x, u)$	right-hand side of nonlinear differential equations in terms of x, u

¹This list is not intended to be all inclusive.

$h(x)$	function of state vector; relative to output function
$\bar{h}(x, u)$	function of state vector and input vector; relative to output function (as in affine systems)
\hat{x}	estimate of a vector
\tilde{x}	error estimate of a vector
$\ x\ $	norm of a vector
$\bar{f}(x)$	function of state vector in affine form of $f(x, u)$
$g(x)$	function in affine form of $f(x, u)$
$\bar{k}(x)$	function in the affine form of $h(x, u)$
k	constant parameter in affine form of $h(x, u)$
J	Jacobian matrix
μ	eigenvalue
$A(t), B(t)$	coefficient matrices in terms of t in system (1.3)
$C(t)$	coefficient in terms of t in output function (1.4)
A, B, C	constant coefficient matrices in observer system (1.6), (1.7)
H	observer gain
H	mapping from input to output (only in Section 4.4)
\bar{F}, G	constant matrices per section 1.3
P, P^{-1}	see sections 1.4, 1.5
h_i	elements of H in section 1.4
$F(t, y, \dot{y})$	first order system as in section 2.2
$U, V, \bar{\rho}, \bar{g}$	defined in section 2.2

$\bar{P}, \bar{Q}, \bar{N}, \bar{C}$	see section 2.2
f, \bar{f}, λ, w	see section 2.3 on canonical DAE observer
v, \underline{g}, Γ	see section 2.3 on canonical DAE observer
D, D^{-1}	matrices as section 2.3
\bar{V}	Lyapunov function as in section 2.4
γ_i	elements of Γ (see section 2.4), of H in section 2.5.1
$\bar{\beta}$	parameter as in section 2.4
$x^*(s)$	Laplace transform of the variable x or $\mathcal{L}[x(t)]$ (see section 2.5.1 and appendix E)
L_p, L_∞, L_2	set of measureable functions as described in section 4.4
$\rho(t)$	parameter as used in section 2.5.1
\bar{D}	region as defined in section 3.2 and Chapter 5
$\bar{A}(t)$	linearizezed matrix as in section 3.2
$Q(t)$	used in coordinate transformation in section 3.2
$\underline{N}(t), \underline{P}(t)$	see representation in section 3.2
$\bar{H}(\hat{x})$	see section 3.2
$\underline{F}(\hat{x}, u)$	see section 3.2
$\Gamma(\hat{x}, u)$	see sections 3.2 and 2.3
R, S	see section 3.2
$\underline{J}(\hat{x}, u)$	see section 3.3
$\bar{G}(\hat{x}, u)$	see section 4.2
$\Sigma_1 \circ \Sigma_2$	composition of subsystems 1 and 2

Appendix F

MATLAB & Simulink Codes

This section contains the programs developed and used by author to produce graphics in Figures F.1, F.2. It should be noted that the programs contain compositions of the programs used to generate graphs in Figures 4.6 - 4.17. Thus, the entire coding can be seen from these programs. Note, functions `diffeqn1` and `diffeqn2` are codes for the ODE systems (4.23), (4.24), respectively; and the functions `odefun1` and `odefun2` are codes for DAE systems (4.23), (4.24), respectively.

The following is the MATLAB Coding for the corresponding
Graphs (Figure F.1 and Figure F.2):

```
global u1;
```

```
global k1;
```

```
u1=1;
```

```
global u2;
```

```
global k2;
```

```
u2=1;
```

```
for k1=-3:3;
```

```
    k1=.1*k1;
```

```
    tspan = [0,1.0];
```

```
    [t,y] = ode45(@diffeqn1,tspan,[0 1.1],[]);
```

```

s11=y(:,1);

s12=y(:,2);

s13=s12-k1*u1;

plot(t,s11,'-',t,s12,'--',t,s13,'k:');

hold on

end

axis([0,.35,-4,4]);

for k2=-3:3;

    k2=.1*k2;

    tspan = [0,1.0];

```



```

[t,y] = ode45(@diffeqn2,tspan,[-1.1 0],[]);

s21=y(:,1);

s22=y(:,2);

s23=s21-k2*u2;

plot(t,s21,'-',t,s22,'--',t,s23,'-');

hold on

end

grid on

hold off

axis([0,.35,-3,2.5]);

```

```

title('Subsystem 1, 2: Solutions using ODE Solver');

text(.01,2,'{ y_1 curves for k_1 for [-.3,.3], u1=1 }');

text(.01,1.5,'{ y_2 curves for k_2 for [-.3,.3], u2=1}');

xlabel('t (time)');

ylabel(' combined graphs for y_1 and y_2');

text(.16,.9,'{y_1} - curves');

text(.16, -2.1,'{y_2} - curves');

%

%function dx = diffeqn1(t,z)

%global u1;

```

```

%global k1;

%dx=zeros(2,1);

%dx(1)=-z(1)+sin(z(2));

%dx(2)=z(2)*u1+z(1);

%

%function dy = diffeqn2(t,y)

%global u2;

%global k2;

%dy(1)=-y(1)+u2*y(2);

%dy(2)=8*exp(y(1))-y(2);

```

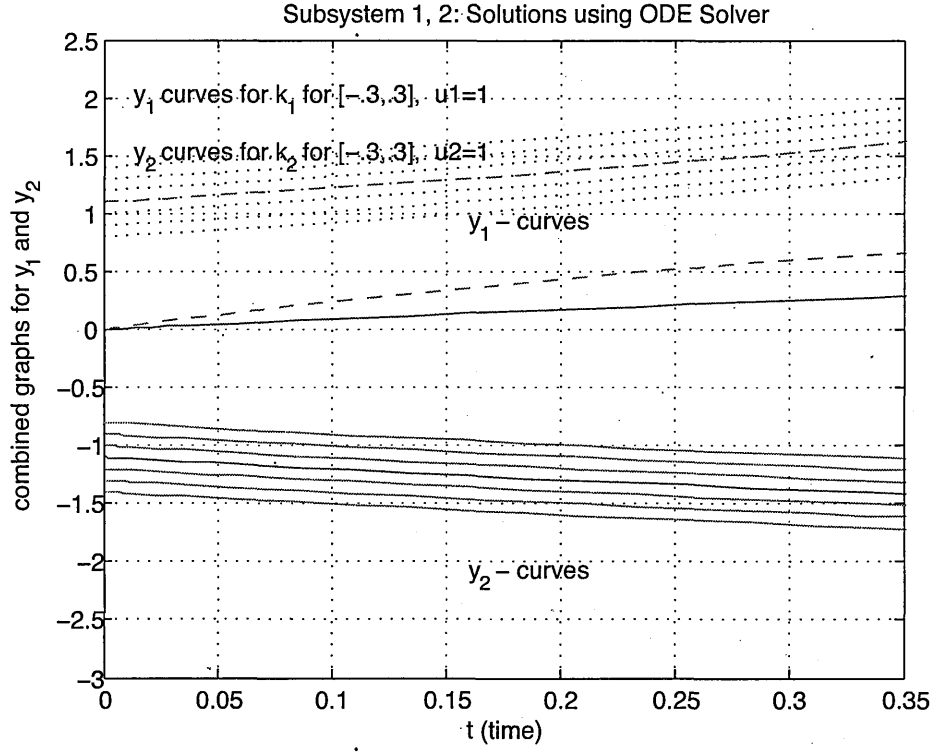


Figure F.1: Graphs of solutions to DAE observer (4.23),(4.24) for system Σ ,

using an ODE solver for input $u_1 = 1$, $u_2 = 1$, parameters $k_1 = [-.3, .3]$,

$k_2 = [-.3, -.3]$, initial conditions $x_1^{(1)}(0) = 0$, $x_2^{(1)}(0) = 1.1$, $y_1(0, u_1) =$

$-k_1 u_1$, $x_1^{(2)}(0) = 0$, $x_2^{(2)}(0) = -1.1$, $y_2(0, u_2) = -k_2 u_2$

```
global u1;
```

```
global k1;
```

```
u1=1;
```

```
global u2;
```

```
global k2;
```

```
u2=1;
```

```
for k1=-3:3;
```

```
    k1=.1*k1;
```

```
    y0=[0;1.1;0];
```

```
    yp0=[0;0;0];
```

```

    tspan = [0,1.0];

    [t,y] = ode15i(@odefun1,tspan,y0,yp0);

    s11=y(:,1);

    s12=y(:,2);

    s13=s12-k1*u1;

    plot(t,s11,'-',t,s12,'--',t,s13,'k:');

    hold on

end

for k2=-3:3;

    k2=.2*k2;

```

```

y20=[-1.1;0;0];

y2p0 = [0; 0; 0];

tspan = [0,1.0];

[t,y] = ode15i(@odefun2,tspan,y20,y2p0);

s21=y(:,1);

s22=y(:,2);

s23=s21-k2*u2;

plot(t,s21,'-',t,s22,'--',t,s23,'-');

hold on

end

```

```
grid on
```

```
hold off
```

```
axis([0,.35,-3,2.5]);
```

```
title('Subsystem 1, 2: Solutions using Implicit DAE Solver');
```

```
text(.01,2,'{ y_1 curves for k_1 for [-.3,.3], u1=1 }');
```

```
text(.01,1.5,'{ y_2 curves for k_2 for [-.6,.6], u2=1 }');
```

```
xlabel('t (time)');
```

```
ylabel(' combined graphs for y_1 and y_2');
```

```
text(.16,.9,'{y_1} - curves');
```

```
text(.16, -2.1,'{y_2} - curves');
```



```

%

%function res = odefun1(t,y,yp)

%global u1;

%global k1;

%res = [yp(1)+y(1)-sin((y(2)))-y(3),

%      yp(2)-yp(3)-u1*y(1)-y(2)-y(3),

%      yp(3)];

%

%function res = odefun2(t,y,yp)

%global u2;

```

```

%global k2;

%res = [yp(1)-y(1)-y(2)*u2-yp(3)-y(3),

%      yp(2)-8*exp(y(1))+y(2)-y(3),

%      yp(3)];

```

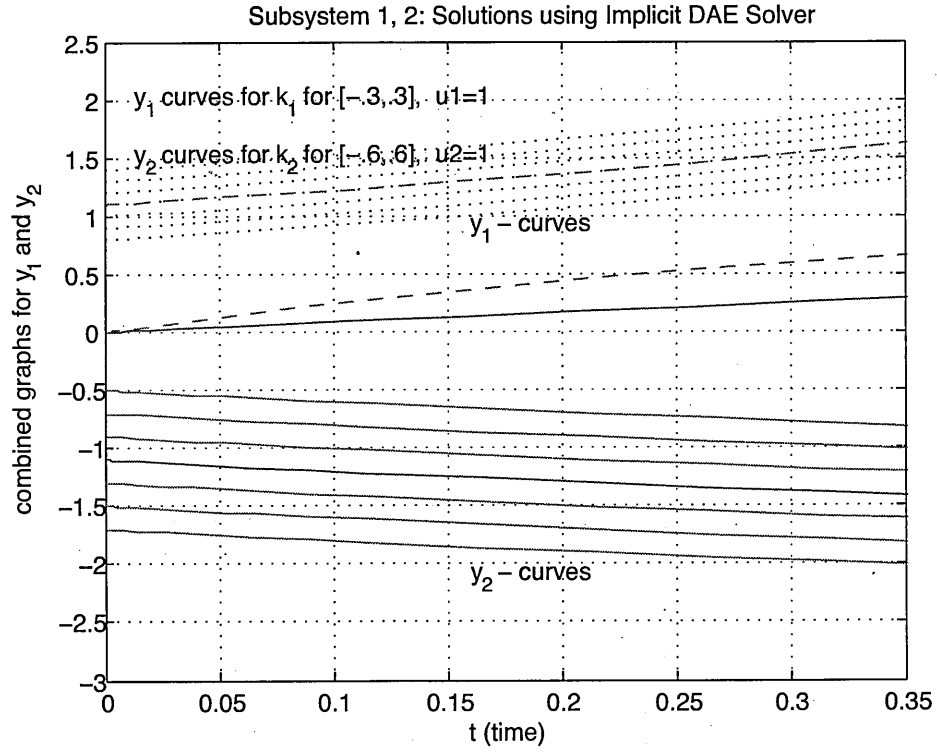


Figure F.2: Solutions to DAE observer (4.23),(4.24) for system Σ , using an

DAE solver for input $u_1 = 1, u_2 = 1$, parameters $k_1 = [-.3, .3], k_2 = [-.6, .6]$,

initial conditions $x_1^{(1)}(0) = 0, x_2^{(1)}(0) = 1.1, y_1(0, u_1) = -k_1 u_1, x_1^{(2)}(0) = 0,$

$x_2^{(2)}(0) = -1.1, y_2(0, u_2) = -k_2 u_2$

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